5 Central Limit Theorem

5. Central Limit Theorem

For $\{Y_i^{N,t}\}$ given by

solution to $i\partial_t \Psi_{N,t} = H_N \Psi_{N,t}$

$$\mathbb{P}_{\Psi_{N,t}}\left[Y_i^{N,t} \in A\right] = \langle \Psi_{N,t}, \mathbb{1}_A(O_i^{(1)})\, \Psi_{N,t} \rangle$$

acting on i-th particle

we analyze

$$\frac{1}{\sqrt{N}} Y^{N,t} := \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(Y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t \rangle \right)$$

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Ex. Factorized state: $Z_i^{N,t}$ random variables with

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$$\mathbb{P}_{\varphi_t^{\otimes N}}[Z_i^{N,t} \in A] = \langle \varphi_t^{\otimes N}, \mathbb{1}_A(O_i^{(1)})\, \varphi_t^{\otimes N} \rangle$$

are i.i.d. Fix $a<b$. Then

$$\mathbb{P}_{\varphi_t^{\otimes N}}\left[\frac{1}{\sqrt{N}} \sum_{i=1}^{N} \left(Z_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t \rangle \right) \in [a,b] \right] \xrightarrow[N\to\infty]{} \mathbb{P}\left[\tilde{\varphi}_t \in [a,b] \right]$$

Gaussian

variance: $\tilde{\sigma}_t = \langle \varphi_t, O^2 \varphi_t \rangle - \langle \varphi_t, O \varphi_t \rangle^2$

THM 5.1 (CLT): Assume $\vartheta^2 \leq c(1-\Delta)$. Fix $a<b$. Then

$$\mathbb{P}_{\Psi_{N,t}}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(y_i^{N,t} - \langle \varphi_t, O^{(1)}\varphi_t\rangle\right) \in [a,b]\right] \xrightarrow{N\to\infty} \mathbb{P}\left[\mathcal{Y}_t \in [a,b]\right]$$

where $\mathcal{Y}_t$ is the centered Gaussian random variable with variance

$$\sigma_t^2 = \| \mathcal{L}_{(t,0)}\, q_t\, O^{(1)} \varphi_t \|_2^2$$

and

$$i\partial_s \mathcal{L}_{(t;s)} = \left(-\Delta + v * |\varphi_s|^2 + q_s K_{1,s} q_s + \overline{q_s} K_{2,s} q_s J\right) \mathcal{L}_{(t;s)}$$

THM 5.1 (CLT): Assume $v \leq C(1-\Delta)$. Fix $a<b$. Then

$$\mathbb{P}_{\Psi_{N,t}}\left[\frac{1}{\sqrt{N}}\sum_{i=1}^{N}\left(Y_i^{N,t} - \langle \varphi_t, O^{(1)}\varphi_t\rangle\right) \in [a,b]\right] \xrightarrow{N\to\infty} \mathbb{P}\left[\mathcal{Y}_t \in [a,b]\right]$$

where $\mathcal{Y}_t$ is the centered Gaussian random variable with variance

$$q_t = 1 - |\varphi_t\rangle\langle\varphi_t|$$

$$\sigma_t^2 = \| \mathcal{L}_{(t,0)}\, q_t\, O^{(1)} \varphi_t \|_2^2$$

and

$$K_{1,s}(x,y) = v(x-y)\varphi_s(x)\overline{\varphi_s(y)}$$

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RMK:

(i) Rate of convergence: $C_{a,b}\, N^{-\frac{1}{2}}\, e^{c|t|}$

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$\rightsquigarrow$ blackboard

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(iii) Symplectic Bogoliubov dynamics

Alternatively

$$\sigma_t^2 = \| \mathcal{L}(t,0)\, q_t O^{(1)} \varphi_t \|_2^2 = \| \Theta(t;0) \left(q_t O^{(1)} \varphi_t , \overline{q_t O^{(1)} \varphi_t} \right) \|_{L^2 \oplus L^2}^2$$

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with $\Theta(t;t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$i\partial_s \Theta(t;s) = \begin{pmatrix} -\Delta + v * |\varphi_s|^2 + q_s K_{1,s} q_s & -\overline{q_s} K_{2,s} q_s \\ \overline{q_s K_{2,s} q_s} & \Delta - v * |\varphi_s|^2 - q_s K_{1,s} q_s \end{pmatrix} \Theta(t;s)$$

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since

$$S = \Theta^*_{(t;s)}\, S\, \Theta_{(t;s)} \qquad \text{with} \qquad S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\Theta(t;s)$ is called symplectic.

LITERATURE

(i) MEAN-FIELD REGIME:

- Ben Arous – Kirkpatrick – Schlein (2013),
  Buchholz – Saffirio – Schlein (2015)
- R. (2022)

(ii) TOWARDS GROSS-PITAEVSKII REGIME

- R. (2020)
- Caraci – Oldenburg – Schlein (2023)

PROOF OF THM. 5.1 follows from (see Ben Arous - Kirkpatrick - Schlein, R.)

PROPOSITION 5.2: Under the same assumptions as in THM 5.1

$$\left| \mathbb{E}_{\Psi_{N,t}} \left[e^{i N^{-\frac{1}{2}} \sum_i \left(y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t \rangle \right)} \right] - e^{-\sigma_t^2/2} \right| \leq C N^{-\frac{1}{2}} e^{e^{C|t|}}$$

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Proof of Prop. 5.2 following R. (2022), Lemm - R. (2023).

IDEA OF THE PROOF:

based on fluctuation dynamics

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$$= e^{-itH_N t}\varphi_0^{\otimes N}$$

$$= \mathcal{U}_{N,t}^* \mathcal{U}_{N,t} e^{-itH_N t} \mathcal{U}_{N,0} \Omega$$

IDEA OF THE PROOF :

based on fluctuation dynamics

$$\mathbb{E}_{\Psi_{N,t}}\left[e^{iN^{-\frac{1}{2}}\sum\left(Y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t\rangle\right)} \right] \quad = d\Gamma\left(O^{(1)} - \langle \varphi_t, O^{(1)}\varphi_t \rangle \right)$$

$$= \langle \Psi_{N,t}, e^{iN^{-\frac{1}{2}}\overbrace{\sum\left(O_i^{(1)} - \langle \varphi_t, O^{(1)} \varphi_t\rangle\right)}} \Psi_{N,t} \rangle$$

$$= e^{-itH_N t} \varphi_0^{\otimes N}$$

$$= \mathcal{U}_{N,t}^* \, \mathcal{U}_{N,t} \, e^{-itH_N t} \, \mathcal{U}_{N,0} \, \Omega$$

$$= \mathcal{U}_{N,t}^* \, \mathcal{W}_N(t;0) \, \Omega$$

IDEA OF THE PROOF:

based on fluctuation dynamics

$$\mathbb{E}_{\Psi_{N,t}}\left[e^{iN^{-\frac{1}{2}}\sum_i \left(y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t\rangle\right)} \right]$$

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$$\tilde{\ } = O^{(1)} - \langle \varphi_t, O \varphi_t \rangle$$

$$= \left\langle W_N(t;0)\Omega,\ \mathcal{U}_{N,t}\, e^{iN^{-\frac{1}{2}} d\Gamma(\tilde{O}^{(1)})}\, \mathcal{U}_{N,t}^{*}\, W_N(t;0)\Omega \right\rangle$$

IDEA OF THE PROOF:

based on fluctuation dynamics

$$\mathbb{E}_{\Psi_{N,t}}\left[e^{iN^{-\frac{1}{2}} \sum_i \left(Y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t \rangle \right)} \right]$$

$$= \left\langle \Psi_{N,t}, \; e^{iN^{-\frac{1}{2}} \sum_i \left(O_i^{(1)} - \langle \varphi_t, O^{(1)} \varphi_t \rangle \right)} \; \Psi_{N,t} \right\rangle$$

$$\sim \; = O^{(1)} - \langle \varphi_t, O \varphi_t \rangle$$

$$= \Big\langle W_N(t;0)\Omega, \; \underbrace{\mathcal{U}_{N,t} \, e^{iN^{-\frac{1}{2}} d\Gamma(\tilde{O}^{(1)})} \, \mathcal{U}_{N,t}^*}_{= e^{iN^{-\frac{1}{2}} \mathcal{U}_{N,t} d\Gamma(\tilde{O}^{(1)}) \mathcal{U}_{N,t}^*}} \; W_N(t;0)\Omega \Big\rangle$$

IDEA OF THE PROOF:

based on fluctuation dynamics

$$\mathbb{E}_{\Psi_{N,t}}\left[e^{iN^{-\frac{1}{2}} \sum (Y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t \rangle)} \right]$$

$$= \langle \Psi_{N,t}, e^{iN^{-\frac{1}{2}} \sum (O_i^{(1)} - \langle \varphi_t, O^{(1)} \varphi_t \rangle)} \Psi_{N,t} \rangle$$

$$\sim = O^{(1)} - \langle \varphi_t, O \varphi_t \rangle$$

$$= \langle W_N(t;0)\Omega, \mathcal{U}_{N,t} e^{iN^{-\frac{1}{2}} d\Gamma(\tilde{O}^{(1)})} \mathcal{U}_{N,t}^* W_N(t;0)\Omega \rangle$$

$$= e^{iN^{-\frac{1}{2}} \mathcal{U}_{N,t} d\Gamma(\tilde{O}^{(1)}) \mathcal{U}_{N,t}^*}$$

$$= e^{iN^{-\frac{1}{2}} d\Gamma(q_t \tilde{O}^{(1)} q_t) + i\phi_+(q_t O \varphi_t)}$$

IDEA OF THE PROOF:

based on fluctuation dynamics

$$\mathbb{E}_{\Psi_{N,t}}\Big[e^{iN^{-\frac{1}{2}}\sum (Y_i^{N,t} - \langle \varphi_t, O^{(1)}\varphi_t\rangle)}\Big]$$

$$= \langle \Psi_{N,t}, e^{iN^{-\frac{1}{2}}\sum (O_i^{(1)} - \langle \varphi_t, O^{(1)}\varphi_t\rangle)} \Psi_{N,t}\rangle$$

$\tilde{O}^{(1)} = O^{(1)} - \langle \varphi_t, O\varphi_t\rangle$

$$= \langle W_N(t;0)\Omega, \mathcal{U}_{N,t}\, e^{iN^{-\frac{1}{2}} d\Gamma(\tilde{O}^{(1)})}\, \mathcal{U}_{N,t}^* W_N(t;0)\Omega\rangle$$

$$= \langle W_N(t;0)\Omega, e^{iN^{-\frac{1}{2}} d\Gamma(q_t\tilde{O}^{(1)}q_t) + i\,\phi_+(q_t\tilde{O}^{(1)}\varphi_t)} W_N(t;0)\Omega\rangle$$

IDEA OF THE PROOF:

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$$= \left\langle \Psi_{N,t},\ e^{iN^{-\frac{1}{2}}\sum\left(Y_i^{N,t} - \langle \varphi_t, O^{(1)}\varphi_t\rangle\right)}\ \Psi_{N,t} \right\rangle$$

$$\tilde{O}^{(1)} = O^{(1)} - \langle \varphi_t, O\varphi_t\rangle$$

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STEP 1

$\approx$ blackboard $\left\langle W_N(t;0)\Omega,\ e^{i\,\phi_+(q_t\tilde{O}^{(1)}\varphi_t)}\ W_N(t;0)\Omega \right\rangle$

IDEA OF THE PROOF:

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$$= \langle W_N(t;0)\Omega, e^{iN^{-\frac{1}{2}} \cancel{d\Gamma(q_t \tilde{O}^{(1)} q_t)} + i\,\phi_+(q_t \tilde{O}^{(1)}\varphi_t)} W_N(t;0)\Omega\rangle$$

STEP 1

$$\approx \langle W_N(t;0)\Omega, e^{i\phi_+(q_t\tilde{O}^{(1)}\varphi_t)} W_N(t;0)\Omega\rangle$$

$$i\partial_t W_N(t;0) = \mathcal{L}_N(t)\, W_N(t;0)$$

$$\approx \langle W_\infty(t;0)\Omega, e^{i\phi_+(q_t\tilde{O}^{(1)}\varphi_t)} W_\infty(t;0)\Omega\rangle$$

$$i\partial_t W_\infty(t;0) = \mathbb{H}\, W_\infty(t;0)$$

IDEA OF THE PROOF:

based on fluctuation dynamics

$$\mathbb{E}_{\Psi_{N,t}}\left[e^{iN^{-\frac12}\sum_i\left(Y_i^{N,t}-\langle\varphi_t,O^{(1)}\varphi_t\rangle\right)}\right]$$

$$=\left\langle \Psi_{N,t},\ e^{iN^{-\frac12}\sum_i\left(Y_i^{N,t}-\langle\varphi_t,O^{(1)}\varphi_t\rangle\right)}\Psi_{N,t}\right\rangle$$

$$\tilde{\ } = O^{(1)}-\langle\varphi_t,O\varphi_t\rangle$$

$$=\left\langle \mathcal{W}_N(t;0)\Omega,\ \mathcal{U}_{N,t}\, e^{iN^{-\frac12}d\Gamma(\tilde O^{(1)})}\,\mathcal{U}_{N,t}^*\,\mathcal{W}_N(t;0)\Omega\right\rangle$$

$$=\left\langle \mathcal{W}_N(t;0)\Omega,\ e^{iN^{-\frac12}\cancel{d\Gamma(q_t\tilde O^{(1)}q_t)}+i\,\phi_+(q_t\tilde O^{(1)}\varphi_t)}\,\mathcal{W}_N(t;0)\Omega\right\rangle$$

STEP 1

$$\approx\left\langle \mathcal{W}_N(t;0)\Omega,\ e^{i\phi_+(q_t\tilde O^{(1)}\varphi_t)}\,\mathcal{W}_N(t;0)\Omega\right\rangle$$

$$i\partial_t\mathcal{W}_N(t;0)=\mathcal{L}_N(t)\,\mathcal{W}_N(t;0)$$

$$\approx\left\langle \mathcal{W}_\infty(t;0)\Omega,\ e^{i\phi_+(q_t\tilde O^{(1)}\varphi_t)}\,\mathcal{W}_\infty(t;0)\Omega\right\rangle$$

$$i\partial_t\mathcal{W}_\infty(t;0)=\mathbb{H}\,\mathcal{W}_\infty(t;0)$$

$\rightsquigarrow$ blackboard

IDEA OF THE PROOF :

based on fluctuation dynamics

$$\mathbb{E}_{\Psi_{N,t}}\left[e^{iN^{-\frac{1}{2}}\sum\left(Y_i^{N,t} - \langle \varphi_t, O^{(1)}\varphi_t\rangle\right)} \right]$$

$$= \left\langle \Psi_{N,t}, \, e^{iN^{-\frac{1}{2}}\sum\left(Y_i^{N,t} - \langle \varphi_t, O^{(1)}\varphi_t\rangle\right)} \Psi_{N,t} \right\rangle$$

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$$= \left\langle W_N(t;0)\Omega, \, e^{iN^{-\frac{1}{2}} \cancel{d\Gamma(q_t \tilde{O}^{(1)} q_t)} + i\,\phi_+(q_t \tilde{O}^{(1)}\varphi_t)} \, W_N(t;0)\Omega \right\rangle$$

STEP 1
$$\approx \left\langle W_N(t;0)\Omega, \, e^{i\phi_+(q_t \tilde{O}^{(1)}\varphi_t)} \, W_N(t;0)\Omega \right\rangle$$

STEP 2
$$\approx \left\langle W_\infty(t;0)\Omega, \, e^{i\phi_+(q_t \tilde{O}^{(1)}\varphi_t)} \, W_\infty(t;0)\Omega \right\rangle$$

$$\approx \left\langle \Omega, \, e^{i\phi_+(\mathcal{L}_{(t;0)} q_t \tilde{O}^{(1)}\varphi_t)} \, \Omega \right\rangle$$

IDEA OF THE PROOF:

based on fluctuation dynamics

$$\mathbb{E}_{\Psi_{N,t}}\left[e^{iN^{-\frac{1}{2}}\sum_i \left(Y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t\rangle\right)} \right]$$

$$= \langle \Psi_{N,t},\ e^{iN^{-\frac{1}{2}}\sum_i \left(Y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t\rangle\right)}\ \Psi_{N,t} \rangle$$

$$\tilde{\ } = O^{(1)} - \langle \varphi_t, O \varphi_t \rangle$$

$$= \langle W_N(t;0)\Omega,\ \mathcal{U}_{N,t}\ e^{iN^{-\frac{1}{2}} d\Gamma(\tilde{O}^{(1)})}\ \mathcal{U}_{N,t}^*\ W_N(t;0)\Omega \rangle$$

$$= \langle W_N(t;0)\Omega,\ e^{iN^{-\frac{1}{2}} \cancel{d\Gamma(q_t \tilde{O}^{(1)} q_t)} + i\phi_+(q_t \tilde{O}^{(1)} \varphi_t)}\ W_N(t;0)\Omega \rangle$$

STEP 1

$$\approx \langle W_N(t;0)\Omega,\ e^{i\phi_+(q_t \tilde{O}^{(1)} \varphi_t)}\ W_N(t;0)\Omega \rangle$$

STEP 2

$$\approx \langle W_\infty(t;0)\Omega,\ e^{i\phi_+(q_t \tilde{O}^{(1)} \varphi_t)}\ W_\infty(t;0)\Omega \rangle$$

$$\approx \langle \Omega,\ e^{i\phi_+(\mathcal{L}_{(t;0)} q_t \tilde{O}^{(1)} \varphi_t)}\ \Omega \rangle$$

STEP 3

$$\approx e^{-\|\mathcal{L}_{(t;0)} q_t \tilde{O}^{(1)} \varphi_t\|^2/2}$$

REMARKS ON CLT

- Random variables $\{Y_i^{N,t}\}$ given by

solution to $i\partial_t \Psi_{N,t} = H_N \Psi_{N,t}$

$$\mathbb{P}_{\Psi_{N,t}}\left[Y_i^{N,t} \in A\right] = \langle \Psi_{N,t}, \mathbb{1}_A(O_i^{(1)})\, \Psi_{N,t} \rangle$$

acting on i-th particle

satisfy CLT.

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- Limiting Gaussian determined by symplectic Bogoliubov dynamics

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- Proof based on

$$\mathcal{W}_N^*(t;0)\,(\mathcal{N}+1)^2\,\mathcal{W}_N(t;0) \leq C_1 e^{C_2|t|}$$

(generalization of Lemma 4.3)

- Proof based on

$$\mathcal{W}_N(t;0)\,\Omega \approx \mathcal{W}_\infty(t;0)\,\Omega$$

with

$$i\partial_t \mathcal{W}_\infty(t;0) = \widetilde{\mathbb{H}}\, \mathcal{W}_\infty(t,0)$$

and

$$\widetilde{\mathbb{H}} = d\Gamma\left(-\Delta + v * |\varphi_t|^2 + q_t K_{1,t} q_t\right) + \frac{1}{2}\iint \left[\left(\overline{q_t} K_{2,t} q_t\right)(x,y)\, a_x^* a_y^* + h.c.\right] dx\,dy$$

called Bogoliubov approximation.

- Proof based on

$$\mathcal{W}_N(t;0)\,\Omega \approx \mathcal{W}_\infty(t;0)\,\Omega$$

with

quadratic in a^*, a

$$i\partial_t \mathcal{W}_\infty(t;0) = \widetilde{\mathbb{H}}\, \mathcal{W}_\infty(t,0)$$

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$$\mathcal{W}_N(t;0)\,\Omega \approx \mathcal{W}_\infty(t;0)\,\Omega \qquad (*)$$

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(*) widely studied (see e.g. Lewin-Nam-Schlein, Grillakis-Machedon, Nam-Napiórkowski, Bossmann-Petrat-Pickl-Soffer)

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Then (R. (2020))

$$\mathcal{W}_\infty^*(t;s)\, \underbrace{A\left((f,g)\right)}_{= a^*(f) + a(g)}\, \mathcal{W}_\infty(t;s) = A\left(\Theta(t;s)(f,g)\right).$$

• Proof based on

$$\mathcal{W}_N(t;0)\,\Omega \approx \mathcal{W}_\infty(t;0)\,\Omega \qquad (*)$$

with quadratic in a^*, a

$$i\partial_t \mathcal{W}_\infty(t;0) = \widetilde{\mathbb{H}}\,\mathcal{W}_\infty(t,0)$$

and

$$\widetilde{\mathbb{H}} = d\Gamma\left(-\Delta + v * |\varphi_t|^2 + q_t K_{1,t} q_t\right) + \frac{1}{2}\iint \left[\left(\overline{q_t} K_{2,t} q_t\right)(x,y)\, a_x^* a_y^* + \text{h.c.}\right] dx\,dy$$

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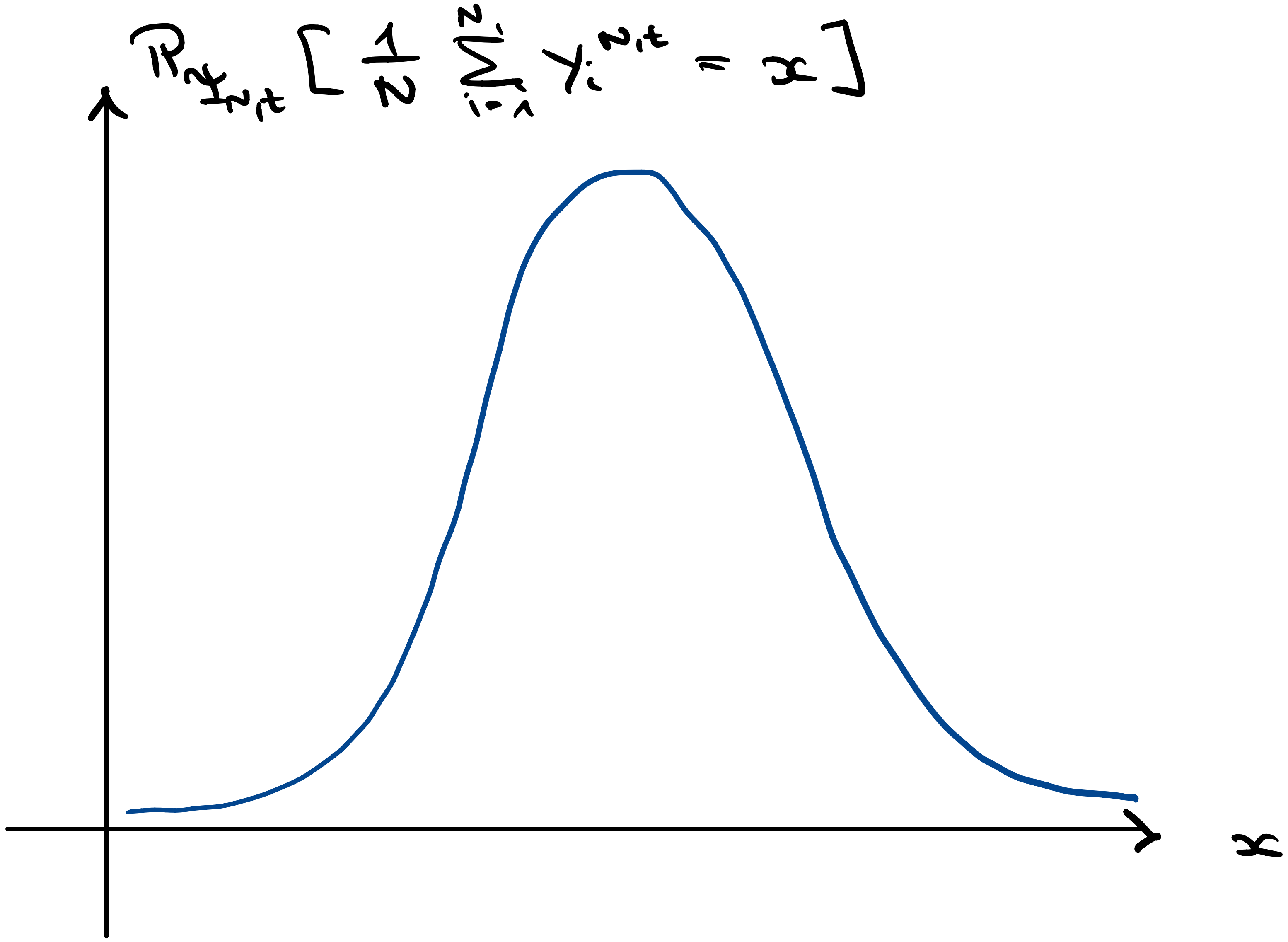
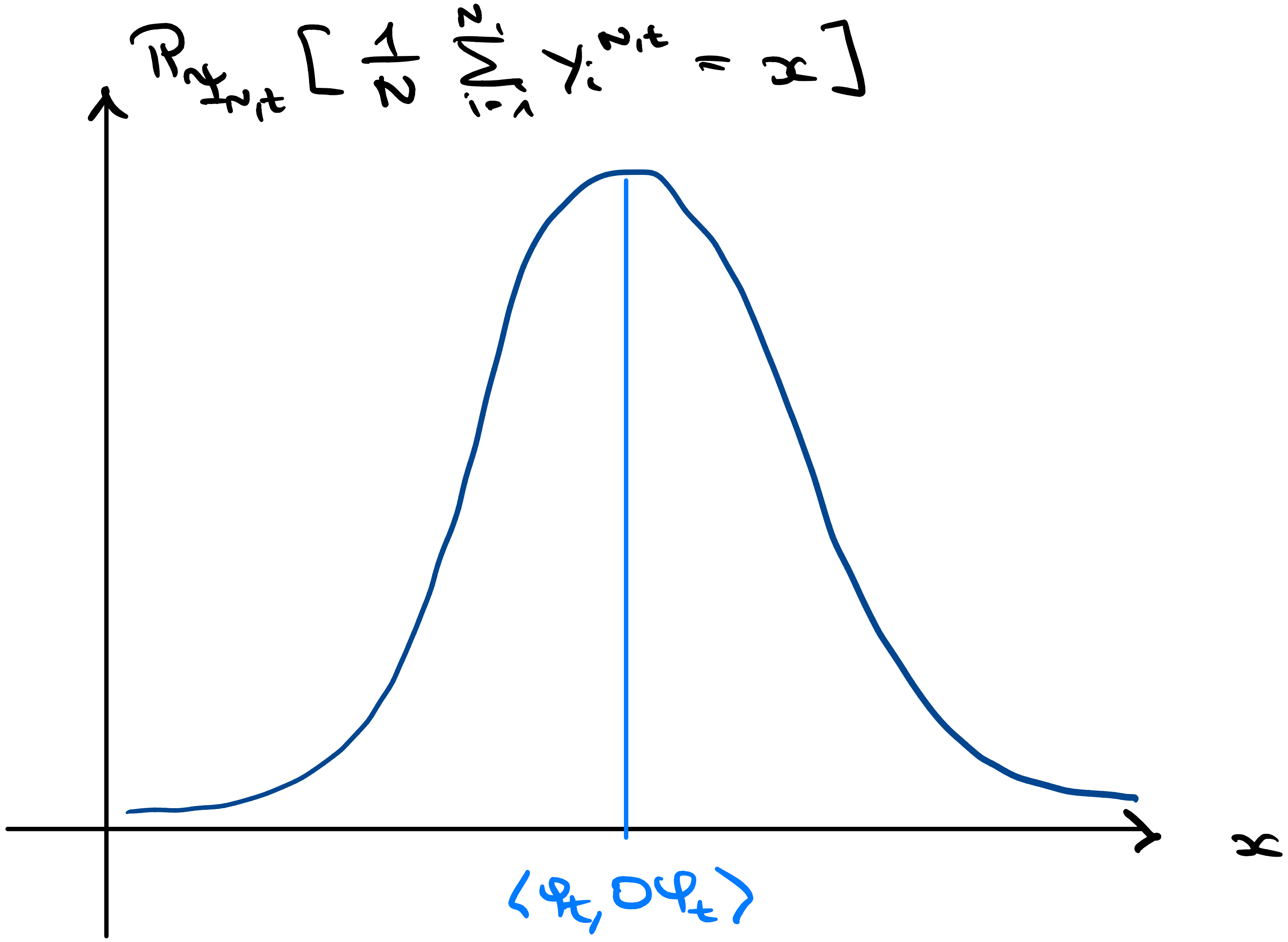
SUMMARY CLT:

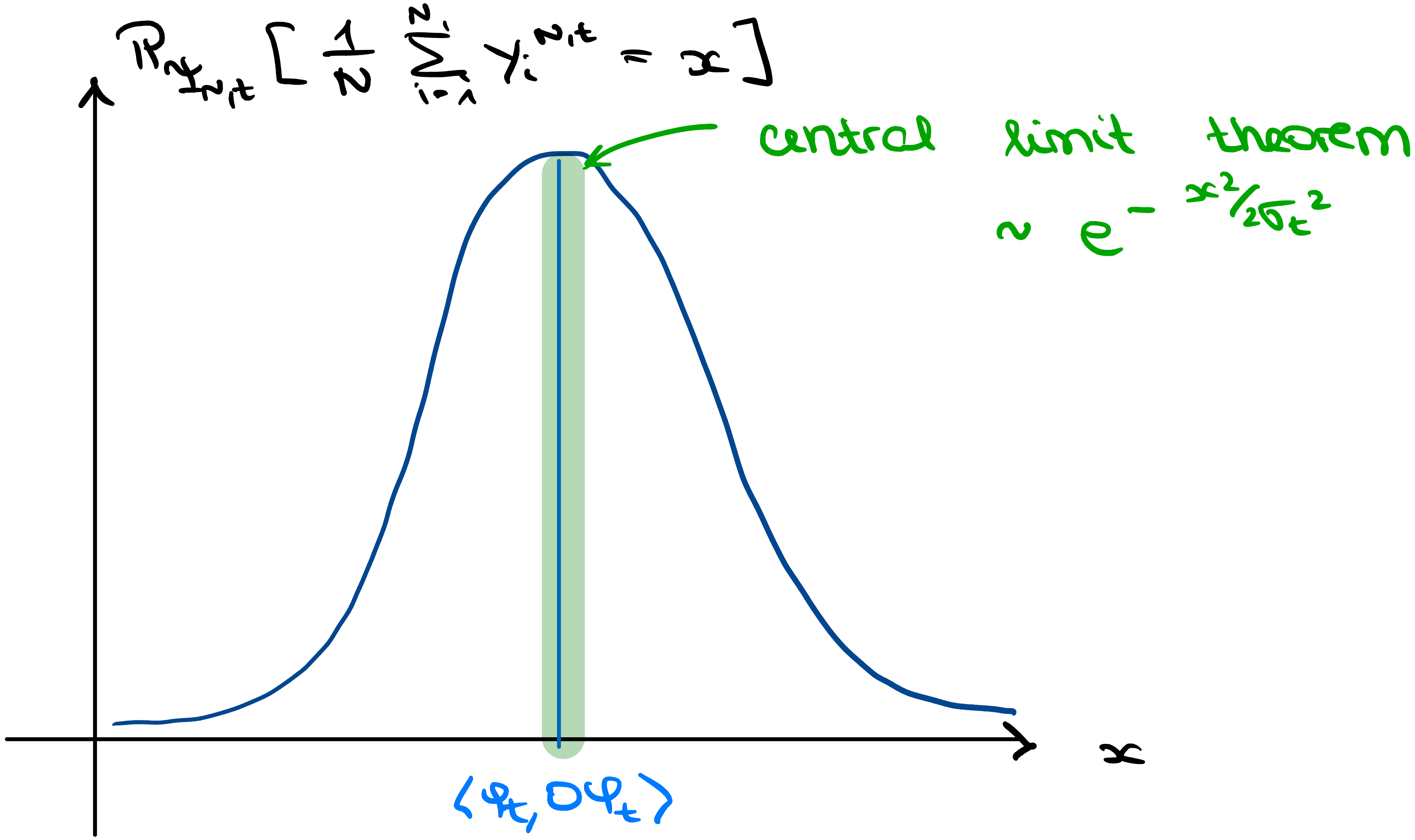


SUMMARY CLT :



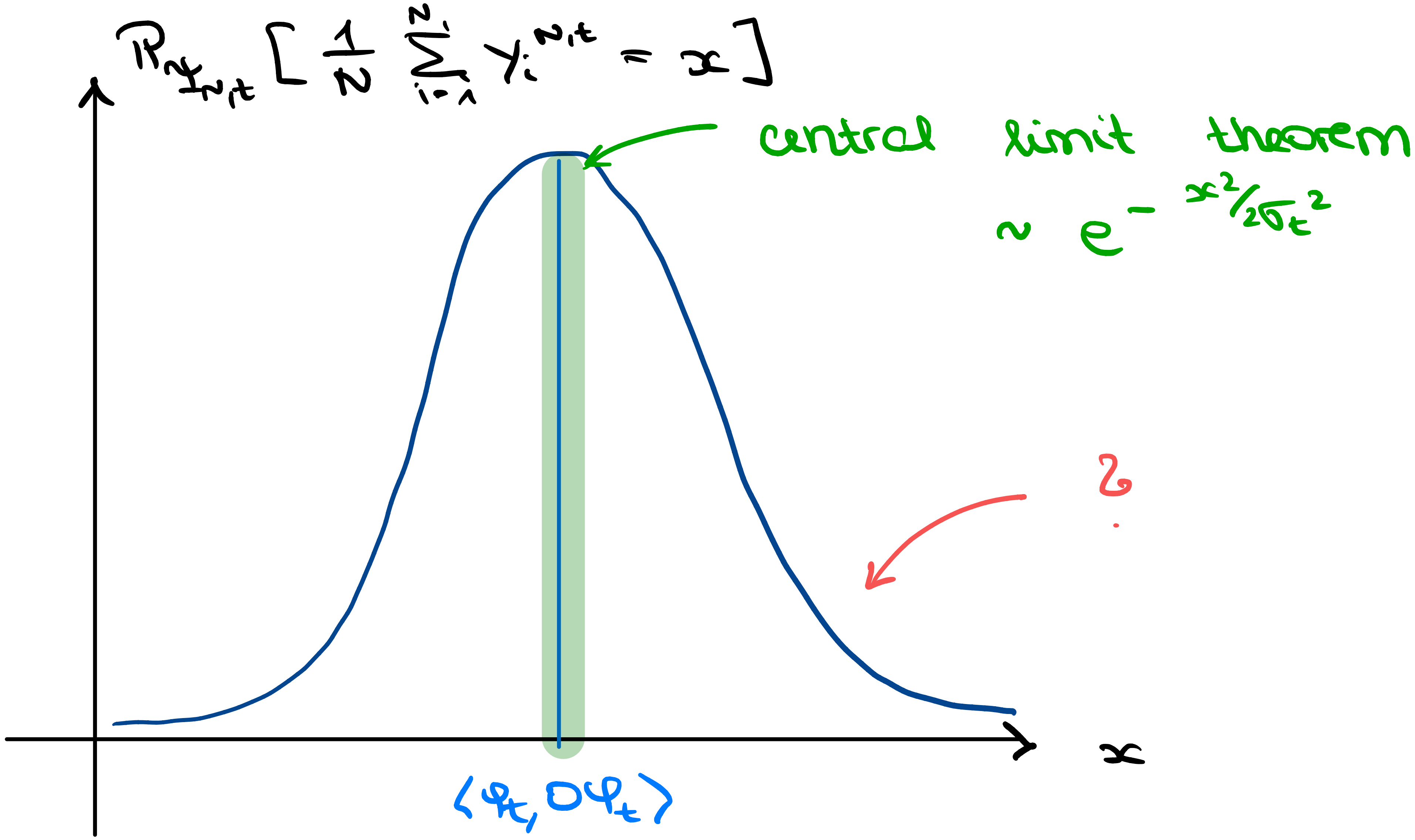
$\mathbb{P}_{\Psi_{N,t}}\left[\frac{1}{N}\sum_{i=1}^{N} y_i^{N,t} = x\right]$
x
$\langle \varphi_t, O\varphi_t \rangle$

SUMMARY CLT:



$\mathbb{P}_{\Psi_{N,t}}\left[\frac{1}{N}\sum_{i=1}^{N} y_i^{N,t} = x\right]$
central limit theorem
$\sim e^{-x^2/2\sigma_t^2}$
x
$\langle \varphi_t, O\varphi_t \rangle$

SUMMARY CLT :



NEXT : Characterization of tails through Large deviations.