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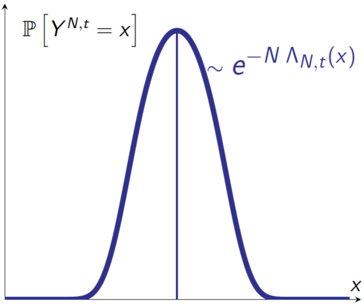

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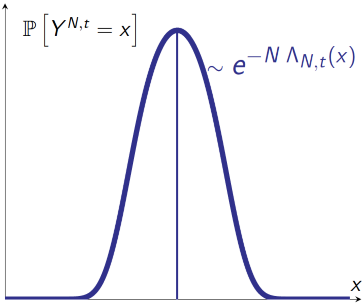

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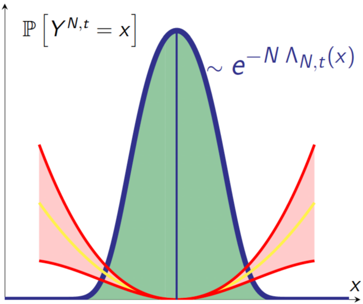

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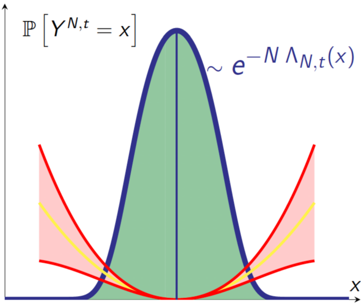

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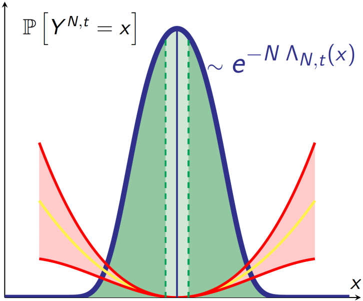

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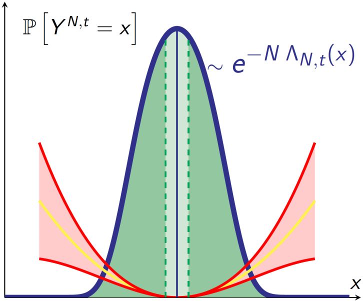

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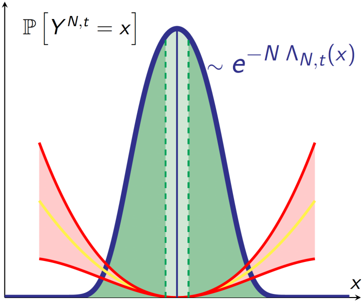

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Step 2: Proof of upper and lower bound.

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Ingredients for the proof:

- Baker-Campbell Hausdorff formulas for non-commuting operators

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Random variables Y_i^N: with law

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for bounded self.adjoint operator O are **identically distributed** and **correlated**.

7.1 Limit theorems

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Law of large numbers: follows from BEC

$$\lim_{N\to\infty} \mathbb{P}_{\psi_N}\left[\left|\frac{1}{N}\sum_{i=1}^{N}\left(Y_i^N - \langle\varphi, O\varphi\rangle\right)\right| > \delta\right] = 0$$

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$$\Lambda_{\psi_N}(x) := -N^{-1} \log\, \mathbb{P}_{\psi_N}\left[N^{-1} \sum_{i=1}^{N} \left(Y_i^N - \langle \varphi, O\varphi \rangle \right) > x \right] .$$

R '23: There exists, $C_1, C_2 > 0$ such that

$$\limsup_{N\to\infty} \Lambda_{\psi_N}(x) \leq \frac{x^2}{2\sigma^2} + C_1 x^3, \quad \liminf_{N\to\infty} \Lambda_{\psi_N}(x) \geq \frac{x^2}{2\sigma^2} - C_2 x^{5/2}$$

- Variance σ^2 agrees with CLTs
- For O such that $\|(-\Delta+1)O(-\Delta+1)^{-1}\|_{\mathrm{op}} \leq C$
- For $\beta = 0$ only

For $O = qOq$: and $\beta = 1$, it holds **R-Nam '23**

$$-\limsup_{N\to\infty} \log \mathbb{P}_{\psi_N}\left[\sum_{i=1}^{N} Y_i^N - \mu_0 > x \right] \leq \frac{x^2}{\sigma_0^2} + Cx^3 .$$

7.3 Empirical measures

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For any $A \subset \mathcal{B}(\mathbb{R})$ compare

$$\nu_N(A) := \frac{1}{N} \sum_{i=1}^{N} \delta_{Y_i^N}(A) \quad \text{and} \quad \nu_\varphi(A) := \langle \varphi,\ \mathbb{1}_A(O)\varphi \rangle_{L^2(\pi)} \ .$$

using **Wasserstein metric $\mathcal{W}_1$**.

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Thanks for your attention.