3 Law of Large Numbers

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$v^2 \leq C(1-\Delta)$, φ_t sol. to Hartree eq. with φ_0.

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For every $\delta > 0$

solution to Schrödinger eq.

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IDEA OF THE PROOF: convergence of reduced particle densities

THM 3.2 (CONVERGENCE OF REDUCED DENSITIES)

Same assumptions as before.
There exist constants $C_1, C_2 > 0$ such that

$$\operatorname{Tr} \left| \gamma^{(1)}_{\Psi_{N,t}} - N \, |\varphi_t\rangle\langle\varphi_t| \right| \leq C_1 e^{C_2 |t|} \qquad (*)$$

for all $N \in \mathbb{N}$ and $t \in \mathbb{R}$.

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- Consequently,

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for fixed $k \in \mathbb{N}$ and $t \in \mathbb{R}$

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- convergence $(*)$ called Bose-Einstein condensation

LITERATURE:

(i) BBGKY hierachy:
- Hepp (1974)
- Ginibre – Velo (1979)
- Spohn (1981)
- Erdős – Yau (2001)
- Bardos – Golse – Mauser (2000)

(ii) New approach
- Rodnianski – Schlein (2009)
- Chen – Lee – Schlein (2011)
- Knowles – Pickl (2010)
- Erdős – Schlein (2009)
- Dietze – Lee (2023)

(iii) Gross – Pitaevskii regime
- Erdős – Schlein – Yau (2006)
- Benedikter – de Oliveira – Schlein (2014)
- Chen – Holmer (2014)
- Brennecke – Schlein (2019)

PROOF OF THM 3.1 FROM THM 3.2

Blackboard.

SUMMARY

- Random variables $\{Y_i^{N,t}\}$ satisfy

$$\mathbb{P}_{\Psi_{N,t}}\left[\left|\frac{1}{N}\sum_{i=1}^{N} Y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t \rangle\right| > \delta\right] \xrightarrow{N\to\infty} 0$$

solution to
$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t}$

solution to
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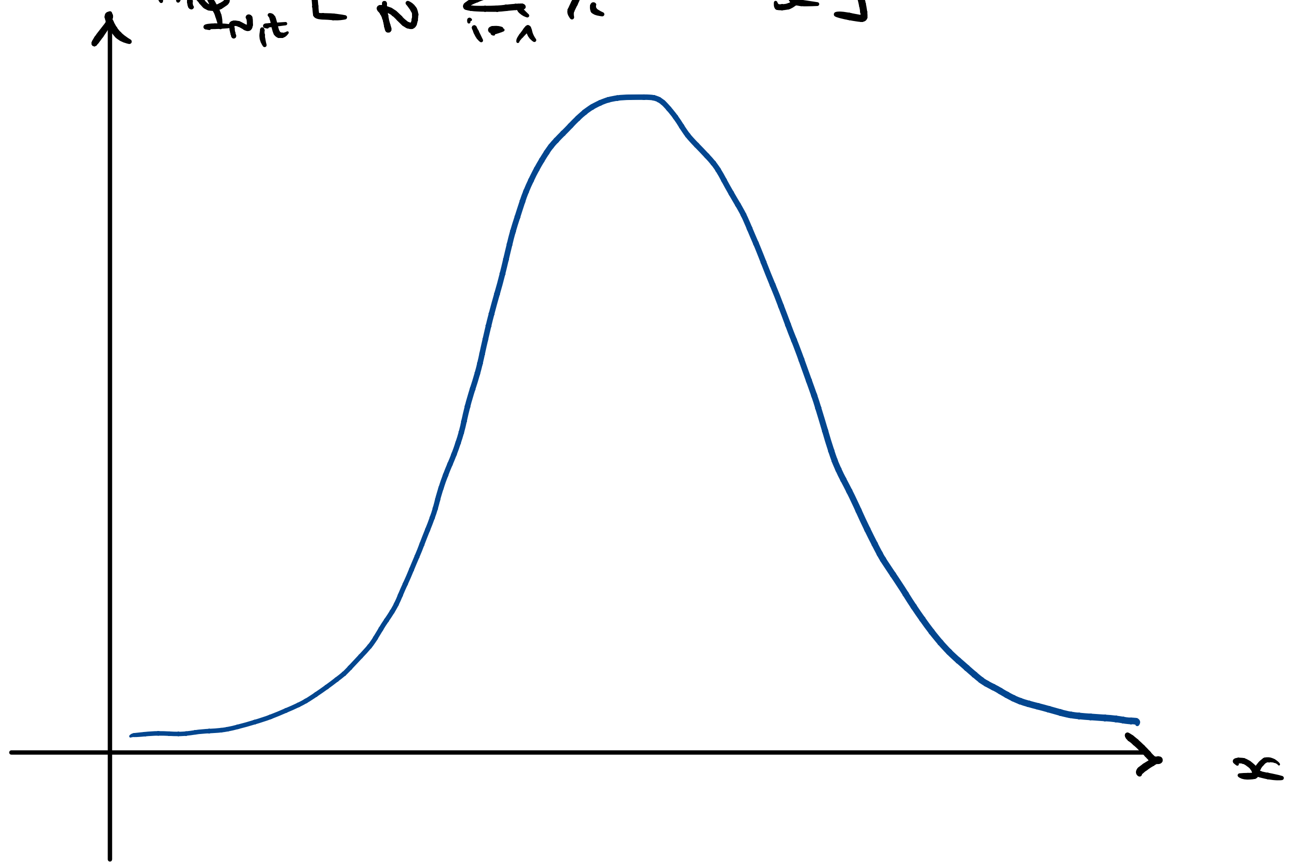
$$\mathbb{P}_{\Psi_{N,t}}\left[\frac{1}{N} \sum_{i=1}^{N} Y_i^{N,t} = x \right]$$



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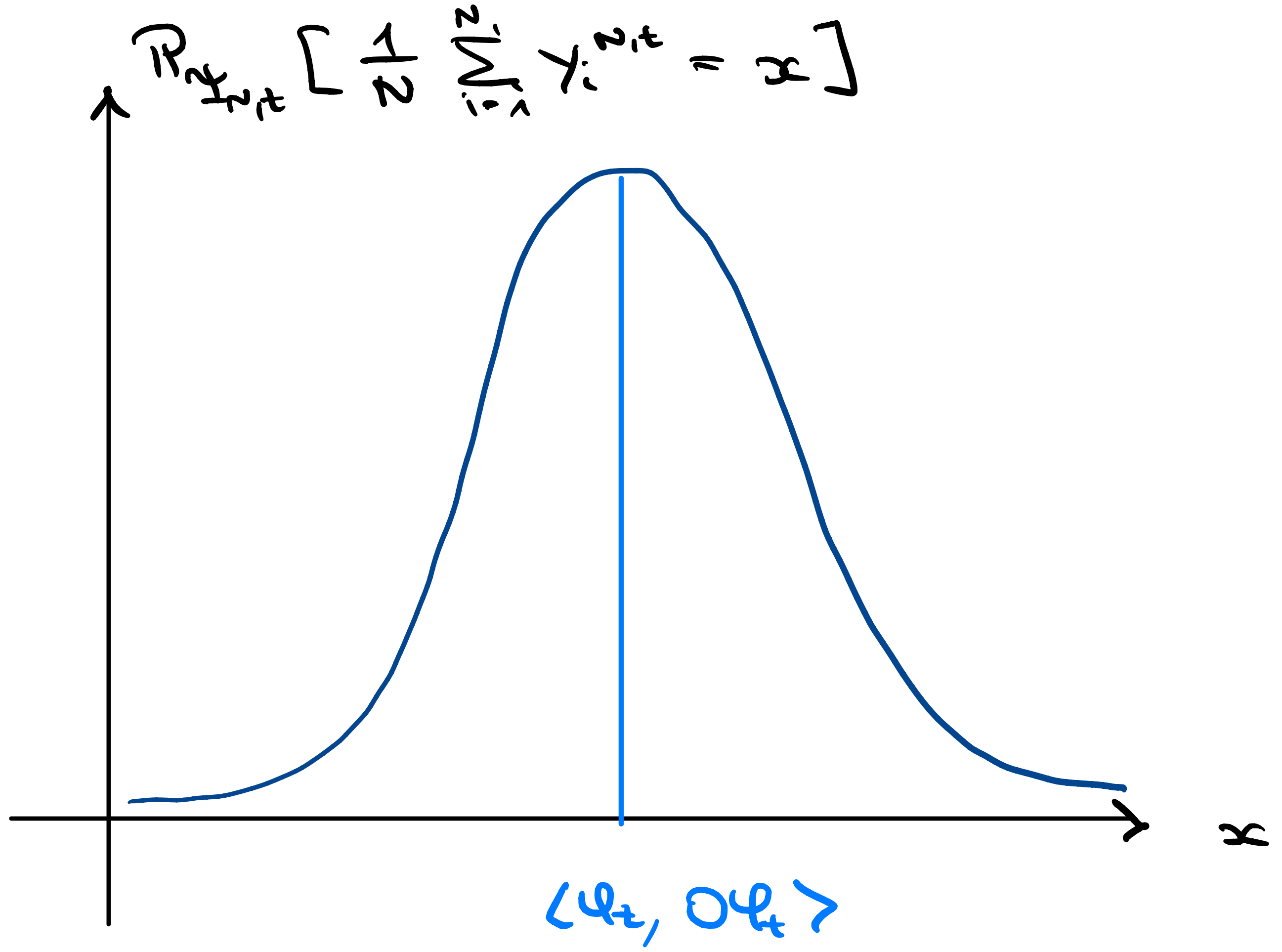

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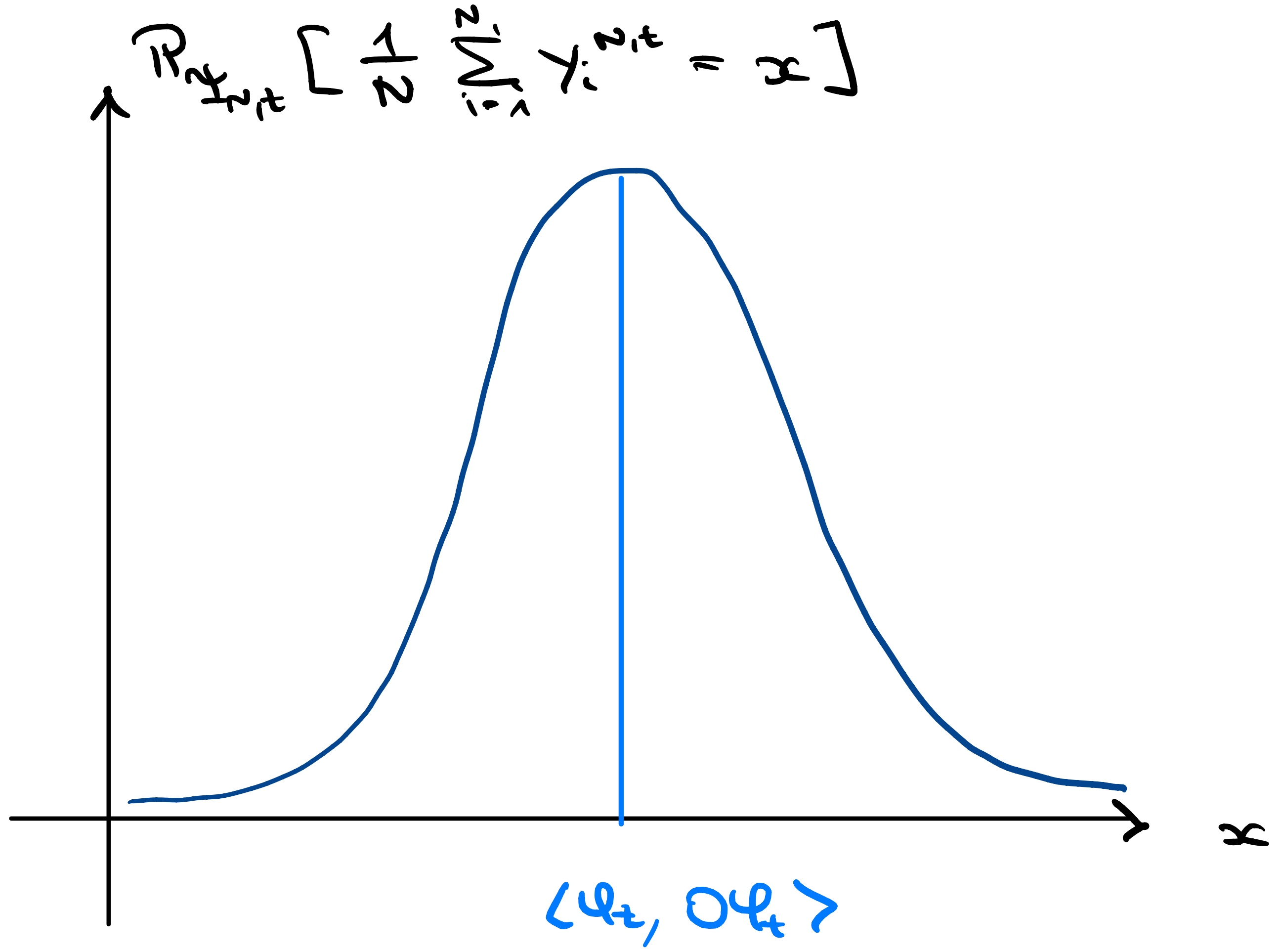

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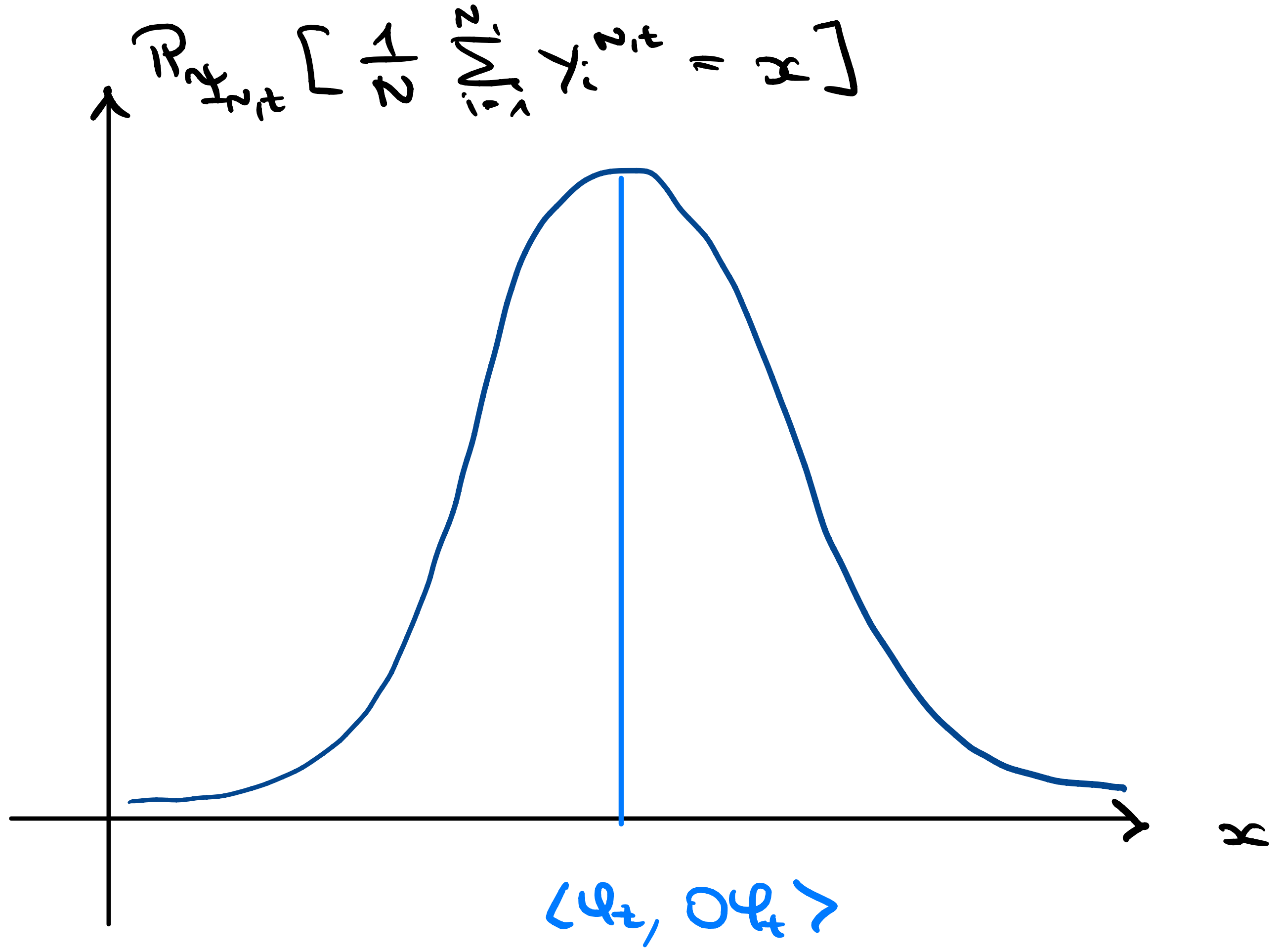

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NEXT : PROOF OF THM 3.2

4 QUANTUM FLUCTUATIONS (AROUND THE CONDENSATE)

4. Fluctuations around the Condensate

4.1 Excitation map

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For $\varphi_t \in L^2(\mathbb{R}^3)$, any $\Psi \in L^2(\mathbb{R}^3)$ has decomposition

$$\Psi = \underbrace{\xi_t^{(0)}}_{\in \mathbb{C}} \varphi_t + \xi_t^{(1)} \qquad \text{with} \quad \xi_t^{(1)} \in L^2_s(\mathbb{R}^3) \quad \& \quad \langle \xi_t^{(1)}, \varphi_t \rangle = 0$$

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$$\Psi = \sum_{j=0}^{1} \xi_t^{(j)} \otimes_s \varphi_t^{(1-j)} \quad \text{with} \quad \xi_t^{(j)} \in L_s^2(\mathbb{R}^{3j}) \quad \& \quad \langle \xi_t^{(j)}, \varphi_t^{\otimes j} \rangle = 0$$

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For $\varphi_t \in L^2(\mathbb{R}^3)$, any $\Psi_N \in L^2(\mathbb{R}^{3N})$ has decomposition

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There exist a unitary

$$\mathcal{U}_{N,t}: \quad \Psi_N \longmapsto \{\xi_t^{(0)}, \dots, \xi_t^{(N)}\} =: \xi_{N,t}$$

$$L_s^2(\mathbb{R}^{3N}) \longrightarrow \bigoplus_{n=0}^{N} L^2_{\perp\varphi_t}(\mathbb{R}^3)^{\otimes_s n}$$

orthogonal compl. of φ_t in $L^2(\mathbb{R}^3)$

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4.2 FOCK SPACE FORMALISM

- Bosonic Fock space is given by

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^3)^{\otimes_s n} \quad , \quad \Psi = \{\psi^{(1)}, \psi^{(2)}, \dots\} \in \mathcal{F}.$$

equipped with

$$\langle \Psi, \Phi \rangle_{\mathcal{F}} = \sum_{n=0}^{\infty} \langle \psi^{(n)}, \varphi^{(n)} \rangle$$

and corresponding norm

$$\| \Psi \|_{\mathcal{F}} = \sum_{n=0}^{\infty} \underbrace{\| \psi^{(n)} \|_2^2}_{\text{prob. that } \Psi \text{ has } n \text{ particles}} .$$

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- Number of particles operator $\mathcal{N}$ defined by

$$\mathcal{N} \psi^{(n)} = n \psi^{(n)}$$

Ex

- N-particle wavefunctions:

$$\Psi = \{0, \dots, 0, \underbrace{\psi_N}_{\sim \text{ N-th component}}, 0 \dots\} \in \mathcal{F}$$

satisfies

$$\mathcal{N}\,\Psi = N\,\Psi$$

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satisfies

$$\mathcal{N}\Psi = N\Psi$$

- vacuum state:

$$\Omega = \{1, 0 \dots\} \in \mathcal{F}$$

satisfies

$$\mathcal{N}\Omega = 0 \cdot \Omega = 0$$

i.e. that is the eigenstate of $\mathcal{N}$ with eigenvalue 0.

- **Creation and annihilation operators:**

For $f \in L^2(\mathbb{R}^3)$ defined by

$$\big(a^*(f)\underbrace{\Psi}_{(n-1)\text{-particle fct}}\big)^{(n)}(x_1,\dots,x_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \underbrace{f(x_j)\,\Psi^{(n-1)}(x_1,\dots,x_{j-1},x_{j+1},\dots,x_n)}_{n\text{-particle fct.}}$$

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RMK :
- $a^*(f)$ is adjoint of $a(f)$
- $a^*(f)$ is linear, $a(f)$ anti-linear in f
- For $f,g \in L^2(\mathbb{R}^3)$, satisfy canonical commutation relations

$$\text{(i)}\ [a(f), a^*(g)] = \langle f,g\rangle_{L^2}$$

$$\text{(ii)}\ [a^*(f), a^*(g)] = 0 = [a(f), a(g)]$$

- Operator valued distributions a_x^*, a_x from

$$a^*(f) = \int f(x)\, a_x^*\, dx \quad , \quad a(f) = \int \bar{f}(x)\, a_x\, dx$$

leading to

(i) $[a_x^*, a_y] = \delta(x-y)$

(ii) $[a_x^*, a_y^*] = [a_x, a_y] = 0$

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leads to (blackboard)

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(i) $\| a(f) \Psi \| \leq \| f \|_2 \, \| \mathcal{N}^{1/2} \Psi \|$

(ii) $\| a^*(f) \Psi \| \leq \| f \|_2 \, \| (\mathcal{N}+1)^{1/2} \Psi \|$

- Observables :
  - We define second quantization $d\Gamma(O^{(1)})$ on $\mathcal{F}$ by

$$\left(d\Gamma(O^{(1)}) \Psi \right)^{(n)} = \sum_{i=1}^{n} O_i^{(1)} \Psi^{(n)}$$

n - particle sector

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- Then

$$\langle \Psi, d\Gamma(O^{(1)})\Phi\rangle = \int dx\,dy\; O^{(1)}(x;y)\,\langle a_x \Phi, a_y \Psi\rangle$$

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$$\langle \Psi, d\Gamma(O^{(1)}) \Phi \rangle = \int dx\, dy\, O^{(1)}(x;y) \langle a_x \Phi, a_y \Psi \rangle$$

leads to

$$d\Gamma(O^{(1)}) = \int O^{(1)}(x;y)\, a_x^* a_y\, dx\, dy$$

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- We define second quantization $d\Gamma(O^{(1)})$ on $\mathcal{F}$ by

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- **Observables**:

- We define second quantization $d\Gamma(O^{(k)})$ on $\mathcal{F}$ by

$$\left(d\Gamma(O^{(k)})\Psi\right)^{(n)} = \sum_{i_1,\dots,i_k}^{n} O^{(k)}_{i_1,\dots,i_k} \Psi^{(n)}$$

- Then

$$\langle \Psi, d\Gamma(O^{(k)})\Phi\rangle = \int dx\,dy\; O^{(k)}(x_1,\dots,x_k,y_1,\dots,y_k) \times \langle a_{x_1}\cdots a_{x_k}\Psi, a_{y_1}\cdots a_{y_k}\Phi\rangle$$

leads to

$$d\Gamma(O^{(k)}) = \int O^{(k)}(x_1,\dots,x_k,y_1,\dots,y_k)\, a^*_{x_1}\cdots a^*_{x_k} a_{y_1}\cdots a_{y_k}\, dx_1\cdots dx_k\, dy_1\cdots dy_k$$

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- Representation of H_N on $\mathcal{F}$

Recall on $L^2_s(\mathbb{R}^{3N})$

$$H_N = \sum_{i=1}^{N} (-\Delta_{x_i}) + \frac{1}{N} \sum_{1 \le i < j < N} v(x_i - x_j)$$

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NEXT: Prove THM 3.2 based on Fock space formalism and $\omega_N(t;0)$ on $\mathcal{F}_{\perp \varphi_t}^{\le N}$.

4.3 PROOF OF THM 3.2

For

$$\operatorname{Tr} \left| \gamma^{(1)}_{\tilde{\Psi}_{N,t}} - N|\varphi_t\rangle\langle\varphi_t| \right| \leq C_1 e^{C_2|t|}$$

it suffices to show

$$\left| \operatorname{Tr} O^{(1)} \left(\gamma^{(1)}_{\tilde{\Psi}_{N,t}} - N|\varphi_t\rangle\langle\varphi_t| \right) \right| \leq C_1 e^{C_2|t|}$$

for all $O^{(1)}$ st. $\|O^{(1)}\|_{op} < \infty$.

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$$\operatorname{Tr} \gamma_{\Psi_{N,t}} O^{(1)} = \langle \Psi_{N,t}, \sum_{i=1}^{N} O^{(1)}_i \underbrace{\Psi_{N,t}}_{= e^{-itH_N t}\varphi^{\otimes N}} \rangle$$

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For this

$N - \mathcal{N}$

$$\mathcal{U}_{N,t}\, a^*(\varphi_t)\, a(\varphi_t)\, \mathcal{U}_{N,t}^* = N - \mathcal{N}_+(t)$$

$$\mathcal{U}_{N,t}\, a^*(f)\, a(g)\, \mathcal{U}_{N,t}^* = a(f)\, a(g) \qquad f, g \in L^2_{\perp \varphi_t}(\mathbb{R}^3)$$

For this

$$\mathcal{U}_{N,t}\, \overbrace{a^*(\varphi_t)}^{\simeq N^{\frac{1}{2}}}\, \overbrace{a(\varphi_t)}^{N^{\frac{1}{2}}}\, \mathcal{U}_{N,t}^* = N - \overbrace{\mathcal{N}_+(t)}^{\mathcal{O}(1) \quad \text{f. } N \to \infty}$$

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$$\mathcal{U}_{N,t}\, a^*(f)a(\varphi_t)\,\mathcal{U}_{N,t}^* = a^*(f)\sqrt{N-\mathcal{N}_+(t)}$$

$$\mathcal{U}_{N,t}\, a^*(\varphi_t)a(g)\,\mathcal{U}_{N,t}^* = \sqrt{N-\underbrace{\mathcal{N}_+(t)}_{O(1)}}\; a(g)$$

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$$\mathcal{U}_{N,t}\, a^*(\varphi_t)\,a(g)\,\mathcal{U}_{N,t}^* = \sqrt{N-\mathcal{N}_+(t)}\; a(g) = \sqrt{N}\, b(g)$$

For this

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- modified creation and annihilation operators $b^*(f), b(g)$

on $\mathcal{F}^{\leq N}_{\perp\varphi_t}$ satisfy

$$[b(g), b^*(f)] = \langle g, f\rangle \left(1 - \frac{\mathcal{N}_+(t)}{N}\right) - \frac{a^*(f)a(g)}{N}$$

For this

$$\mathcal{U}_{N,t}\, a^*(\varphi_t) a(\varphi_t)\, \mathcal{U}_{N,t}^* = N - \mathcal{N}_+(t)$$

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$$[b(g), b^*(f)] = \langle g, f \rangle \left(1 - \cancel{\frac{\mathcal{N}_+(t)}{N}}\right) - \cancel{\frac{a^*(f) a(g)}{N}}$$

i.e. $\quad b(g), b^*(f) \approx a(g), a^*(f) \quad$ for $N \to \infty$

and

$$\| b(g) \psi \| \leq \| \mathcal{N}^{1/2} \psi \| , \quad \| b^*(f) \psi \| \leq \| (\mathcal{N}+1) \psi \|$$

for all $\psi \in \mathcal{F}^{\leq N}_{\perp\varphi_t}$.

(see Brennecke - Schlein).

With this

$$\operatorname{Tr} \gamma_{\Psi_{N,t}} O^{(1)} = \langle W_N(t;0)\Omega, \underbrace{\mathcal{U}_{N,t}\, d\Gamma(O^{(1)})\, \mathcal{U}_{N,t}^*}_{\text{blackboard}} W_N(t;0)\Omega\rangle$$

With this

$$\mathrm{Tr}\,\gamma_{\tilde{\Psi}_{N,t}} O^{(1)} = \langle W_N(t;0)\Omega,\ \mathcal{U}_{N,t}\, d\Gamma(O^{(1)})\, \mathcal{U}_{N,t}^* \, W_N(t;0)\Omega\rangle$$

$$\begin{aligned} &= N\,\langle \varphi_t, O\varphi_t\rangle \\ &\quad + \sqrt{N}\,\langle W_N(t;0)\Omega,\ \overbrace{\phi_+(q_t O\varphi_t)}^{= b(q_t O\varphi_t) + b^*(q_t O\varphi_t)}\ W_N(t;0)\Omega\rangle \\ &\quad + \langle W_N(t;0)\Omega,\ \big(d\Gamma(q_t O q_t) - \langle \varphi_t, O\varphi_t\rangle \mathcal{N}_+(t)\big)\, W_N(t;0)\Omega\rangle \end{aligned}$$

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$$\operatorname{Tr} \gamma_{\Psi_{N,t}} O^{(1)} = \langle \mathcal{W}_N(t;0)\Omega, \mathcal{U}_{N,t}\, d\Gamma(O^{(1)})\, \mathcal{U}_{N,t}^* \, \mathcal{W}_N(t;0)\Omega \rangle$$

$$= N \langle \varphi_t, O \varphi_t \rangle + \sqrt{N} \langle \mathcal{W}_N(t;0)\Omega, \underbrace{\phi_+(q_t O \varphi_t)}_{= b(q_t O \varphi_t) + b^*(q_t O \varphi_t)} \mathcal{W}_N(t;0)\Omega \rangle + \langle \mathcal{W}_N(t;0)\Omega, \left(d\Gamma(q_t O q_t) - \langle \varphi_t, O \varphi_t \rangle \mathcal{N}_+(t) \right) \mathcal{W}_N(t;0)\Omega \rangle$$

leading to

$$\left| \operatorname{Tr} \gamma_{\Psi_{N,t}} O^{(1)} - N \langle \varphi_t, O \varphi_t \rangle \right| \leq \left| \langle \mathcal{W}_N(t;0)\Omega, \left(d\Gamma(q_t O q_t) - \langle \varphi_t, O \varphi_t \rangle \mathcal{N}_+(t) \right) \mathcal{W}_N(t;0)\Omega \rangle \right| + \sqrt{N} \left| \langle \mathcal{W}_N(t;0)\Omega, \phi_+(q_t O \varphi_t) \mathcal{W}_N(t;0)\Omega \rangle \right|$$

With this

$$\mathrm{Tr}\, \gamma_{\Psi_{N,t}} O^{(1)} = \langle \mathcal{W}_N(t;0)\Omega, \mathcal{U}_{N,t}\, d\Gamma(O^{(1)})\, \mathcal{U}_{N,t}^* \,\mathcal{W}_N(t;0)\Omega\rangle$$

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leading to

$$\left|\mathrm{Tr}\, \gamma_{\Psi_{N,t}} O^{(1)} - N\langle \varphi_t, O\varphi_t\rangle\right|$$
$$\leq \left| \langle \mathcal{W}_N(t;0)\Omega, \left(d\Gamma(q_t O q_t) - \langle \varphi_t, O\varphi_t\rangle \mathcal{N}_+(t)\right) \mathcal{W}_N(t;0)\Omega\rangle\right.$$
$$\left. + \sqrt{N} \left|\langle \mathcal{W}_N(t;0)\Omega, \phi_+(q_t O\varphi_t) \mathcal{W}_N(t;0)\Omega\rangle\right|\right.$$

$$\leq C \langle \mathcal{W}_N(t;0)\Omega, \mathcal{N}\, \mathcal{W}_N(t;0)\Omega\rangle$$
$$+ C\sqrt{N} \langle \mathcal{W}_N(t;0)\Omega, (\mathcal{N}+1)^{1/2} \mathcal{W}_N(t;0)\Omega\rangle \overset{!}{\leq} C\sqrt{N} e^{c|t|}$$

improved rate: Chen-Lee-Schlein

LEMMA 4.3 Under the same assumptions as in Thm 2, there ex. $c_1, c_2 > 0$ st.

$$\langle \mathcal{W}_N(t;0)\Omega, \mathcal{N}\, \mathcal{W}_N(t;0)\Omega \rangle \leq c_1 e^{c_2|t|} \langle \Omega, (\mathcal{N}+1)\Omega \rangle$$

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LEMMA 4.3 Under the same assumptions as in Thm 2, there ex. $C_1, C_2 > 0$ st.

$$\langle \mathcal{W}_N(t;0)\Omega, \mathcal{N}\, \mathcal{W}_N(t;0)\Omega \rangle \leq C_1 e^{C_2|t|} \langle \Omega, \cancel{(\mathcal{N}+1)}^{=1}\, \Omega \rangle$$

IDEA OF THE PROOF :

Based on

$$i\partial_t \mathcal{W}_N(t;0) = \mathcal{L}_N(t)\, \mathcal{W}_N(t;0) \quad ; \quad \mathcal{W}_N(t;t) = 1.$$

LEMMA 4.3 Under the same assumptions as in Thm 2, there ex. $C_1, C_2 > 0$ st.

$$\langle \mathcal{W}_N(t;0)\Omega, \mathcal{N}\, \mathcal{W}_N(t;0)\Omega \rangle \leq C_1 e^{C_2|t|} \langle \Omega, \underbrace{\cancel{(\mathcal{N}+1)}}_{=1}\Omega \rangle$$

IDEA OF THE PROOF :

Based on

$$i\partial_t \mathcal{W}_N(t;0) = \mathcal{L}_N(t)\, \mathcal{W}_N(t;0) \quad ; \quad \mathcal{W}_N(t;t) = 1.$$

we write

$$\mathcal{W}_N^*(t;0)\, \mathcal{N}\, \mathcal{W}_N(t;0) - \mathcal{N}$$

LEMMA 4.3 Under the same assumptions as in Thm 2, there ex. $C_1, C_2 > 0$ st.

$$\langle \mathcal{W}_N(t;0)\Omega, \mathcal{N}\, \mathcal{W}_N(t;0)\Omega \rangle \leq C_1 e^{C_2|t|} \langle \Omega, \underbrace{\cancel{(\mathcal{N}+1)}}_{=1} \Omega \rangle$$

IDEA OF THE PROOF :

Based on

$$i\partial_t \mathcal{W}_N(t;0) = \mathcal{L}_N(t)\, \mathcal{W}_N(t;0) \quad ; \quad \mathcal{W}_N(t;t) = 1.$$

we write

$$\mathcal{W}_N^*(t;0)\, \mathcal{N}\, \mathcal{W}_N(t;0) - \mathcal{N} = - \mathcal{W}_N^*(t;s)\, \mathcal{N}\, \mathcal{W}_N(t;s) \Big|_{s=0}^{s=t}$$

LEMMA 4.3 Under the same assumptions as in Thm 2, there ex. $c_1, c_2 > 0$ st.

$$\langle \mathcal{W}_N(t;0)\Omega, \mathcal{N}\, \mathcal{W}_N(t;0)\Omega \rangle \leq c_1 e^{c_2|t|} \langle \Omega, \cancel{(\mathcal{N}+1)}\, \Omega \rangle \quad = 1$$

IDEA OF THE PROOF:

Based on

$$i\partial_t \mathcal{W}_N(t;0) = \mathcal{L}_N(t)\, \mathcal{W}_N(t;0) \quad ; \quad \mathcal{W}_N(t;t) = 1. \quad (*)$$

we write

$$\mathcal{W}_N^*(t;0)\, \mathcal{N}\, \mathcal{W}_N(t;0) - \mathcal{N} = -\mathcal{W}_N^*(t;s)\, \mathcal{N}\, \mathcal{W}_N(t;s)\Big|_{s=0}^{s=t}$$

fundamental thm. of calculus

$$= -\int_0^t ds\, \frac{d}{ds}\Big(\mathcal{W}_N^*(t;s)\, \mathcal{N}\, \mathcal{W}_N(t;s) \Big)$$

LEMMA 4.3 Under the same assumptions as in Thm 2, there ex. $C_1, C_2 > 0$ st.

$$\langle \mathcal{W}_N(t;0)\Omega, \mathcal{N}\, \mathcal{W}_N(t;0)\Omega\rangle \leq C_1 e^{C_2|t|} \langle \Omega, \underbrace{(\cancel{\mathcal{N}+1})}_{=1}\Omega\rangle$$

IDEA OF THE PROOF:

Based on

$$i\partial_t \mathcal{W}_N(t;0) = \mathcal{L}_N(t)\, \mathcal{W}_N(t;0) \quad ; \quad \mathcal{W}_N(t;t) = 1. \quad (*)$$

we write

$$\mathcal{W}_N^*(t;0)\, \mathcal{N}\, \mathcal{W}_N(t;0) - \mathcal{N} = -\mathcal{W}_N^*(t;s)\, \mathcal{N}\, \mathcal{W}_N(t;s)\Big|_{s=0}^{s=t}$$

fundamental thm. of calculus

$$= -\int_0^t ds \, \frac{d}{ds}\left(\mathcal{W}_N^*(t;s)\, \mathcal{N}\, \mathcal{W}_N(t;s)\right)$$

$$\overset{(*)}{=} +\int_0^t ds \; \mathcal{W}_N^*(t;s)\left(\mathcal{L}_N(s)\mathcal{N} - \mathcal{N}\mathcal{L}_N(s)\right)\mathcal{W}_N(t;s)$$

LEMMA 4.3 Under the same assumptions as in Thm 2, there ex. $C_1, C_2 > 0$ st.

$$\langle \mathcal{W}_N(t;0)\Omega, \mathcal{N}\, \mathcal{W}_N(t;0)\Omega\rangle \leq C_1 e^{C_2|t|} \langle \Omega, \underbrace{\cancel{(\mathcal{N}+1)}}_{=1} \Omega\rangle$$

IDEA OF THE PROOF:

Based on

$$i\partial_t \mathcal{W}_N(t;0) = \mathcal{L}_N(t)\, \mathcal{W}_N(t;0) \quad ; \quad \mathcal{W}_N(t;t) = 1. \qquad (*)$$

we write

$$\mathcal{W}_N^*(t;0)\, \mathcal{N}\, \mathcal{W}_N(t;0) - \mathcal{N} = -\mathcal{W}_N^*(t;s)\, \mathcal{N}\, \mathcal{W}_N(t;s)\Big|_{s=0}^{s=t}$$

fundamental thm. of calculus

$$= -\int_0^t ds\, \frac{d}{ds}\left(\mathcal{W}_N^*(t;s)\, \mathcal{N}\, \mathcal{W}_N(t;s)\right)$$

$$\overset{(*)}{=} + \int_0^t ds\, \mathcal{W}_N^*(t;s)\left(\mathcal{L}_N(s)\mathcal{N} - \mathcal{N}\mathcal{L}_N(s)\right)\mathcal{W}_N(t;s)$$

$$= \int_0^t ds\, \mathcal{W}_N^*(t;s)\, [\mathcal{L}_N(s), \mathcal{N}]\, \mathcal{W}_N(t;s)$$

LEMMA 4.3 Under the same assumptions as in Thm 2, there ex. $C_1, C_2 > 0$ s.t.

$$\langle \mathcal{W}_N(t;0)\Omega, \mathcal{N}\, \mathcal{W}_N(t;0)\Omega \rangle \le C_1 e^{C_2|t|} \langle \Omega, \cancel{(\mathcal{N}+1)}^{=1} \Omega \rangle$$

<u>IDEA OF THE PROOF</u>:

Based on

$$i\partial_t \mathcal{W}_N(t;0) = \mathcal{L}_N(t)\, \mathcal{W}_N(t;0) \quad ; \quad \mathcal{W}_N(t;t) = 1. \quad (*)$$

we write

$$\mathcal{W}_N^*(t;0)\, \mathcal{N}\, \mathcal{W}_N(t;0) - \mathcal{N} = -\mathcal{W}_N^*(t;s)\, \mathcal{N}\, \mathcal{W}_N(t;s)\Big|_{s=0}^{s=t}$$

fundamental thm. of calculus

$$= -\int_0^t ds\, \frac{d}{ds}\left(\mathcal{W}_N^*(t;s)\, \mathcal{N}\, \mathcal{W}_N(t;s)\right)$$

$$\overset{(*)}{=} + \int_0^t ds\, \mathcal{W}_N^*(t;s)\left(\mathcal{L}_N(s)\mathcal{N} - \mathcal{N}\mathcal{L}_N(s)\right)\mathcal{W}_N(t;s)$$

$$= \int_0^t ds\, \mathcal{W}_N^*(t;s)\, \underbrace{[\mathcal{L}_N(s), \mathcal{N}]}_{\text{GOAL: } \le C\mathcal{N}}\, \mathcal{W}_N(t;s)$$

and conclude by Gronwall.

LEMMA 4.3 Under the same assumptions as in Thm 2, there ex. $C_1, C_2 > 0$ st.

$$\langle \mathcal{W}_N(t;0)\Omega, \mathcal{N}\, \mathcal{W}_N(t;0)\Omega\rangle \leq C_1 e^{C_2|t|} \langle \Omega, \cancel{(\mathcal{N}+1)}\,\Omega\rangle \quad = 1$$

<u>IDEA OF THE PROOF</u>:

Based on

Step 1

$$i\partial_t \mathcal{W}_N(t;0) = \mathcal{L}_N(t)\,\mathcal{W}_N(t;0) \quad ; \quad \mathcal{W}_N(t;t) = 1. \quad (*)$$

we write

$$\mathcal{W}_N^*(t;0)\,\mathcal{N}\,\mathcal{W}_N(t;0) - \mathcal{N} = -\mathcal{W}_N^*(t;s)\,\mathcal{N}\,\mathcal{W}_N(t;s)\Big|_{s=0}^{s=t}$$

fundamental thm. of calculus

$$= -\int_0^t ds\, \frac{d}{ds}\left(\mathcal{W}_N^*(t;s)\,\mathcal{N}\,\mathcal{W}_N(t;s)\right)$$

$$\overset{(*)}{=} + \int_0^t ds\, \mathcal{W}_N^*(t;s)\left(\mathcal{L}_N(s)\,\mathcal{N} - \mathcal{N}\,\mathcal{L}_N(s)\right)\mathcal{W}_N(t;s)$$

$$= \int_0^t ds\, \mathcal{W}_N^*(t;s)\,\underbrace{\left[\mathcal{L}_N(s), \mathcal{N}\right]}_{\text{GOAL: } \leq C\mathcal{N}}\,\mathcal{W}_N(t;s)$$

Step 2

and conclude by Gronwall. (see for ex. Lewin-Nam-Schlein)

PROOF: (OF LEMMA 4.3)

STEP 1: GOAL: $\boxed{i\partial_t \mathcal{W}_N(t;s) = \mathcal{L}_N(t)\mathcal{W}_N(t;s)}$

PROOF: (OF LEMMA 4.3)

STEP 1: GOAL: $\boxed{i\partial_t \mathcal{W}_N(t;s) = \mathcal{L}_N(t)\,\mathcal{W}_N(t;s)}$

We have

$$i\partial_t \mathcal{W}_N(t;s) = i\partial_t \left(\mathcal{U}_{N,t}^* e^{-i\mathcal{H}_N(t-s)} \mathcal{U}_{N,s} \right)$$

PROOF: (OF LEMMA 4.3)

STEP 1: GOAL: $\boxed{i\partial_t \mathcal{W}_N(t;s) = \mathcal{L}_N(t)\mathcal{W}_N(t;s)}$

We have

$$i\partial_t \mathcal{W}_N(t;s) = i\partial_t \left(\mathcal{U}_{N,t}^* e^{-iH_N(t-s)} \mathcal{U}_{N,s} \right)$$

$$= \left[-\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} + \mathcal{U}_{N,t}^* H_N \mathcal{U}_{N,t} \right] \mathcal{U}_{N,t}\, e^{-iH(t-s)} \mathcal{U}_{N,s}$$

PROOF: (OF LEMMA 3)

STEP 1: GOAL: $\boxed{i\partial_t \mathcal{W}_N(t;s) = \mathcal{L}_N(t)\,\mathcal{W}_N(t;s)}$

We have

$$i\partial_t \mathcal{W}_N(t;s) = i\partial_t \left(\mathcal{U}_{N,t}^* e^{-iH_N(t-s)} \mathcal{U}_{N,s} \right)$$

$$= \left[-\left(i\partial_t \mathcal{U}_{N,t}^*\right) \mathcal{U}_{N,t} + \mathcal{U}_{N,t}^* H_N \mathcal{U}_{N,t} \right] \mathcal{U}_{N,t}^* e^{-iH(t-s)} \mathcal{U}_{N,s}$$

$$= \Big[\underbrace{-(\partial_t \mathcal{U}_{N,t}^*)\mathcal{U}_{N,t} + \mathcal{U}_N^* H_N \mathcal{U}_{N,t}}_{\mathcal{L}_N(t)} \Big] \mathcal{W}_N(t;s)$$

Since

$$\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} \sqrt{N}\,\Phi_+(q_t i\partial_t\varphi_t) + a^*(\varphi_t)a(q_t i\partial_t\varphi_t) + \langle\varphi_t, i\partial_t\varphi_t\rangle(N-\mathcal{N}_+(t))$$

since

$$\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} = \sqrt{N}\,\overbrace{\Phi_+(q_t i\partial_t\varphi_t)}^{=b^*(q_t i\partial_t\varphi_t)+b(q_t i\partial_t\varphi_t)} + a^*(\varphi_t)a(q_t i\partial_t\varphi_t) + \langle\varphi_t, i\partial_t\varphi_t\rangle(N-\mathcal{N}_+(t))$$

and

since

$$\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} = \sqrt{N}\,\Phi_t(q_t i\partial_t \varphi_t) + a^*(\varphi_t)\,a(q_t i\partial_t \varphi_t) + \langle \varphi_t, i\partial_t \varphi_t\rangle (N - \mathcal{N}_+(t))$$

$$H_N = \int (-\Delta_x)\, a_x^* a_x \,dx + \frac{1}{2N}\int v(x-y)\, a_x^* a_y^* a_x a_y$$

and

$$\mathcal{U}_{N,t}^* H_N \mathcal{U}_{N,t} =$$

since

prop. $\mathcal{U}_{N,t}$

$$\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} = \sqrt{N}\,\Phi_+(q_t i\partial_t \varphi_t) + a^*(\varphi_t)a(q_t i\partial_t \varphi_t) + \langle \varphi_t, i\partial_t \varphi_t\rangle (N - \mathcal{N}_+(t))$$

and

$$H_N = \int (-\Delta_x) a_x^* a_x \, dx + \frac{1}{2N}\int v(x-y)\, a_x^* a_y^* a_x a_y$$

$$\begin{aligned}
\mathcal{U}_{N,t}^* H_N \mathcal{U}_{N,t} = \; & (N - \mathcal{N}_+(t))\,\langle \varphi_t, (-\Delta + v * |\varphi_t|^2)\varphi_t\rangle \\
& + \sqrt{N}\,\Phi_+(q_t(-\Delta + v*|\varphi_t|^2)\varphi_t) \\
& + d\Gamma(q_t(-\Delta + v*|\varphi_t|^2)q_t) \\
& + \int v(x-y)\,\varphi_t(x)\varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,y})\, dx\,dy \\
& + \frac{1}{2}\int v(x-y)\,\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + \text{h.c.}\right] dx\,dy \\
& + \sum_{i=1}^{3} R_N^{(i)}(t)
\end{aligned}$$

since

$$\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} \underbrace{\sqrt{N}\,\phi_+(q_t i\partial_t \varphi_t)}_{\text{linear}} + \underbrace{a^*(\varphi_t)a(q_t i\partial_t \varphi_t)}_{\text{quadratic}} + \langle \varphi_t, i\partial_t\varphi_t\rangle (N - \mathcal{N}_+(t))$$

and

$$\mathcal{U}_{N,t}^* H_N \mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} (N - \mathcal{N}_+(t))\langle \varphi_t, (-\Delta + v * |\varphi_t|^2)\varphi_t\rangle$$

$$+ \sqrt{N}\,\phi_+(q_t(-\Delta + v*|\varphi_t|^2)\varphi_t) \quad \} \text{ linear}$$

$$\left.\begin{aligned} &+ d\Gamma(q_t(-\Delta + v*|\varphi_t|^2)q_t) \\ &+ \int v(x-y)\varphi_t(x)\varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,y})\, dx\,dy \\ &+ \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + \text{h.c.}\right] dx\,dy \end{aligned}\right\} \text{quadratic}$$

$$+ \sum_{i=1}^{3} R_N^{(i)}(t) \quad \} \text{ cubic + quartic}$$

since

$$\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} \underbrace{\sqrt{N}\,\Phi_+(q_t i\partial_t \varphi_t)}_{\text{linear}} + \underbrace{a^*(\varphi_t)a(q_t i\partial_t \varphi_t)}_{\text{quadratic}} + \langle \varphi_t, i\partial_t \varphi_t\rangle (N-\mathcal{N}_+(t))$$

and

$$\begin{aligned}
\mathcal{U}_{N,t}^* H_N' \mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} \;& (N-\mathcal{N}_+(t))\,\langle \varphi_t, (-\Delta + v*|\varphi_t|^2)\varphi_t\rangle \\
&+ \sqrt{N}\,\Phi_+(q_t(-\Delta + v*|\varphi_t|^2)\varphi_t) \quad \Big\}\ \text{linear} \\
&+ d\Gamma(q_t(-\Delta + v*|\varphi_t|^2)q_t) \\
&+ \int v(x-y)\,\varphi_t(x)\varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,y})\, dx\,dy \quad \Big\}\ \text{quadratic} \\
&+ \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + \text{h.c.}\right] dx\,dy \\
&+ \sum_{i=1}^{3} \mathcal{R}_N^{(i)}(t) \quad \Big\}\ \text{cubic + quartic}
\end{aligned}$$

since

$$\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} \sqrt{N}\,\underbrace{\phi_+(q_t i\partial_t\varphi_t)}_{\text{linear}} + \underbrace{a^*(\varphi_t)\,a(q_t i\partial_t \varphi_t)}_{\text{quadratic}} + \langle\varphi_t, i\partial_t\varphi_t\rangle(N-\mathcal{N}_+(t))$$

and

$$
\begin{aligned}
\mathcal{U}_{N,t}^* H_N \mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} \; & (N-\mathcal{N}_+(t))\,\langle\varphi_t, (-\Delta + v*|\varphi_t|^2)\varphi_t\rangle \\
& + \sqrt{N}\,\phi_+\big(q_t(-\Delta + v*|\varphi_t|^2)\varphi_t\big) \qquad \Big\}\ \text{linear} \\
& \left.\begin{aligned}
& + d\Gamma\big(q_t(-\Delta + v*|\varphi_t|^2)q_t\big) \\
& + \int v(x-y)\,\varphi_t(x)\varphi_t(y)\,a^*(q_{t,x})\,a(q_{t,y})\,dx\,dy \\
& + \frac{1}{2}\int v(x-y)\,\big[\varphi_t(x)\varphi_t(y)\,b^*(q_{t,x})\,b^*(q_{t,y}) + \text{h.c.}\big]\,dx\,dy
\end{aligned}\right\}\ \text{quadratic} \\
& + \sum_{i=1}^{3} R_N^{(i)}(t) \qquad \Big\}\ \text{cubic + quartic}
\end{aligned}
$$

Since

$$\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} \underbrace{\cancel{\sqrt{N}\,\Phi_+(q_t i\partial_t\varphi_t)}}_{\text{linear}} + \underbrace{a^*(\varphi_t)a(q_t i\partial_t\varphi_t)}_{\text{quadratic}} + \langle\varphi_t, i\partial_t\varphi_t\rangle(N-\mathcal{N}_+(t))$$

and

$$i\partial_t\varphi_t = \left(-\Delta + v*|\varphi_t|^2\right)\varphi_t$$

$$\begin{aligned}\mathcal{U}_{N,t}^* H_N \mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} \; & (N-\mathcal{N}_+(t))\,\langle\varphi_t, (-\Delta + v*|\varphi_t|^2)\varphi_t\rangle \\ & \cancel{+\sqrt{N}\,\Phi_+(q_t(-\Delta+v*|\varphi_t|^2)\varphi_t)} \quad \Big\}\ \text{linear} \\ & + d\Gamma\left(q_t(-\Delta + v*|\varphi_t|^2)q_t\right) \\ & + \int v(x-y)\,\varphi_t(x)\varphi_t(y)\,a^*(q_{t,x})\,a(q_{t,y})\,dx\,dy \\ & + \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\,b^*(q_{t,x})\,b^*(q_{t,y}) + \text{h.c.}\right]dx\,dy \quad \Big\}\ \text{quadratic} \\ & + \sum_{i=1}^{3} R_N^{(i)}(t) \quad \Big\}\ \text{cubic + quartic}\end{aligned}$$

since

prop. $\mathcal{U}_{N,t}$

$$\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} = \underbrace{\cancel{\sqrt{N}\,\Phi_+\left(q_t i\partial_t\varphi_t\right)}}_{\text{linear}} + \underbrace{a^*(\varphi_t)\,a(q_t i\partial_t\varphi_t)}_{\text{quadratic}} + \cancel{\langle\varphi_t, i\partial_t\varphi_t\rangle\left(N-\mathcal{N}_+(t)\right)}$$

and

$i\partial_t\varphi_t = \left(-\Delta + v*|\varphi_t|\right)\varphi_t$

prop. $\mathcal{U}_{N,t}$

$$\begin{aligned}
\mathcal{U}_{N,t}^* H_N' \mathcal{U}_{N,t} = \; & \cancel{\left(N-\mathcal{N}_+(t)\right)\left\langle\varphi_t, \left(-\Delta + v*|\varphi_t|^2\right)\varphi_t\right\rangle} \\
& \cancel{+ \sqrt{N}\,\Phi_+\left(q_t\left(-\Delta + v*|\varphi_t|^2\right)\varphi_t\right)} \quad \Big\} \text{ linear} \\
& + d\Gamma\left(q_t\left(-\Delta + v*|\varphi_t|^2\right)q_t\right) \\
& + \int v(x-y)\,\varphi_t(x)\varphi_t(y)\,a^*(q_{t,x})\,a(q_{t,y})\,dx\,dy \\
& + \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\,b^*(q_{t,x})\,b^*(q_{t,y}) + h.c.\right]dx\,dy \\
& + \sum_{i=1}^{3} \mathcal{R}_N^{(i)}(t) \quad \Big\} \text{ cubic + quartic}
\end{aligned}$$

quadratic

since

$$\left(i\partial_t \mathcal{U}_{N,t}^*\right)\mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} \underbrace{\cancel{\sqrt{N}\,\Phi_+(q_+ i\partial_t \varphi_t)}}_{\text{linear}} + \underbrace{a^*(\varphi_t)\,a(q_+ i\partial_t \varphi_t)}_{\text{quadratic}} + \cancel{\langle \varphi_t, i\partial_t \varphi_t\rangle (N - \mathcal{N}_+(t))}$$

and

$$i\partial_t \varphi_t = \left(-\Delta + v * |\varphi_t|^2\right)\varphi_t$$

$$\mathcal{U}_{N,t}^* H_N \mathcal{U}_{N,t} \overset{\text{prop. } \mathcal{U}_{N,t}}{=} \cancel{(N - \mathcal{N}_+(t))\,\langle \varphi_t, (-\Delta + v * |\varphi_t|^2)\varphi_t\rangle}$$

$$+ \cancel{\sqrt{N}\,\Phi_+(q_t(-\Delta + v * |\varphi_t|^2)\varphi_t)} \quad \} \text{ linear}$$

$$\left.\begin{aligned} &+ d\Gamma\left(q_t(-\Delta + v * |\varphi_t|^2)q_t\right) \\ &+ \int v(x-y)\,\varphi_t(x)\varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,y})\, dx\,dy \\ &+ \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + \text{h.c.}\right] dx\,dy \end{aligned}\right\} \text{quadratic}$$

$$+ \sum_{i=1}^{3} R_N^{(i)}(t) \quad \} \text{ cubic + quartic}$$

and on $\mathcal{F}_{\perp\varphi_t}^{\leq N}$

$$-\underbrace{a^*(\varphi_t)\,a(q_+ i\partial_t \varphi_t)} + d\Gamma\left(q_t(-\Delta + v * |\varphi_t|^2)q_t\right) = d\Gamma\left(-\Delta + v * |\varphi_t|^2\right)$$

we have

$$\mathcal{L}_N(t) = \mathbb{H} + \sum_{j=1}^{3} R_N^{(j)}(t)$$

with

$$\mathbb{H} = d\Gamma\left(-\Delta + v * |\varphi_t|^2 + q_t K_1 q_t\right)$$
$$+ \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + h.c.\right] dx\,dy$$

and

$$R_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma\left(q_t\left[v * |\varphi_t|^2 + K_{1,t}\right] q_t\right) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b\left(q_t\left[v * |\varphi_t|^2\right]\varphi_t\right) + h.c.$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}} \int v(x-y)\, \varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\,dy + h.c.$$

$$R_{N,3}(t) = \frac{1}{2N} \int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\,dy$$

we have

$$\mathcal{L}_N(t) = \mathbb{H} + \sum_{j=1}^{3} R_N^{(j)}(t)$$

with

$$K_{1,t}(x,y) = \varphi_t(x)\overline{\varphi_t(y)}\, v(x-y)$$

$$\mathbb{H} = d\Gamma(-\Delta + v * |\varphi_t|^2 + q_t K_1 q_t)$$

$$q_{t,y}(x) = q_t(x,y)$$

$$+ \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + h.c.\right] dx\,dy$$

and

$$R_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma(q_t[v*|\varphi_t|^2 + K_{1,t}]q_t) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b(q_t[v*|\varphi_t|^2]\varphi_t) + h.c.$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}}\int v(x-y)\,\varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\,dy + h.c.$$

$$R_{N,3}(t) = \frac{1}{2N}\int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\,dy$$

we have

remainder (cubic + quartic)

$$\mathcal{L}_N(t) = \mathbb{H} + \sum_{j=1}^{3} \mathcal{R}_N^{(j)}(t)$$

with

$$K_{1,t}(x,y) = \varphi_t(x)\overline{\varphi_t(y)}\, v(x-y)$$

$$\mathbb{H} = d\Gamma\left(-\Delta + v*|\varphi_t|^2 + q_t K_1 q_t\right)$$

$$q_{t,y}(x) = q_t(x,y)$$

$$+ \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + h.c.\right] dx\,dy$$

and

$$\mathcal{R}_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma\left(q_t\left[v*|\varphi_t|^2 + K_{1,t}\right] q_t\right) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b\left(q_t\left[v*|\varphi_t|^2\right]\varphi_t\right) + h.c.$$

$$\mathcal{R}_{N,2}(t) = \frac{1}{\sqrt{N}}\int v(x-y)\,\varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\,dy + h.c.$$

$$\mathcal{R}_{N,3}(t) = \frac{1}{2N}\int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\,dy$$

We have

remainder (cubic + quartic)

$$\mathcal{L}_N(t) = \mathbb{H} + \underbrace{\cancel{\sum_{j=1}^{3} R_N^{(j)}(t)}}_{} \qquad \text{as } N \to \infty$$

with

$$K_{1,t}(x,y) = \varphi_t(x)\overline{\varphi_t(y)}\, v(x-y)$$

$$\mathbb{H} = d\Gamma\left(-\Delta + v * |\varphi_t|^2 + q_t K_1 q_t\right)$$

$$q_{t,y}(x) = q_t(x,y)$$

$$+ \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + h.c.\right] dx\, dy$$

and

$$R_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma\left(q_t\left[v*|\varphi_t|^2 + K_{1,t}\right] q_t\right) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b\left(q_t\left[v*|\varphi_t|^2\right]\varphi_t\right) + h.c.$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}}\int v(x-y)\, \varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\, dy + h.c.$$

$$R_{N,3}(t) = \frac{1}{2N}\int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\, dy$$

We have

remainder (cubic + quartic)

$$\mathcal{L}_N(t) = \underbrace{\mathbb{H}}_{\text{quadratic}}^{\text{leading order}} + \cancel{\sum_{j=1}^{3} R_N^{(j)}(t)} \quad \text{as } N \to \infty$$

with

$$K_{1,t}(x,y) = \varphi_t(x)\overline{\varphi_t(y)}\, v(x-y)$$

$$\mathbb{H} = d\Gamma\left(-\Delta + v * |\varphi_t|^2 + q_t K_1 q_t\right)$$

$$q_{t,y}(x) = q_t(x,y)$$

$$+ \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + h.c.\right] dx\,dy$$

and

$$R_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma\left(q_t\left[v*|\varphi_t|^2 + K_{1,t}\right] q_t\right) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b\left(q_t\left[v*|\varphi_t|^2\right]\varphi_t\right) + h.c.$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}}\int v(x-y)\,\varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\,dy + h.c.$$

$$R_{N,3}(t) = \frac{1}{2N}\int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\,dy$$

We have

remainder (cubic + quartic)

leading order

$$\mathcal{L}_N(t) = \mathbb{H} + \cancel{\sum_{j=1}^{3} R_N^{(j)}(t)} \qquad \text{as } N \to \infty$$

quadratic

with

$K_{1,t}(x,y) = \varphi_t(x)\overline{\varphi_t(y)}\, v(x-y)$

$$\mathbb{H} = d\Gamma\left(-\Delta + v * |\varphi_t|^2 + q_t K_1 q_t\right)$$

$q_{t,y}(x) = q_t(x,y)$

$$+ \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + h.c.\right] dx\,dy$$

and

$$R_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma\left(q_t\left[v*|\varphi_t|^2 + K_{1,t}\right] q_t\right) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b\left(q_t\left[v*|\varphi_t|^2\right]\varphi_t\right) + h.c.$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}} \int v(x-y)\, \varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\,dy + h.c.$$

$$R_{N,3}(t) = \frac{1}{2N} \int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\,dy$$

that finishes step 1.

STEP 2 :

GOAL : $\boxed{ |\langle \Psi, [\mathcal{N}, \mathcal{L}_N(t)] \Psi \rangle| \leq C \langle \Psi, (\mathcal{N}+1) \Psi \rangle \quad \text{f.} \ \Psi \in \mathcal{F}_{\perp \varphi_t}^{\leq N} }$

STEP 2 :

GOAL : $\boxed{\left|\langle \Psi, [\mathcal{N}, \mathcal{L}_N(t)] \Psi \rangle\right| \leq C \langle \Psi, (\mathcal{N}+1) \Psi \rangle \quad \text{f. } \Psi \in \mathcal{F}_{\perp \varphi_t}^{\leq N}}$

$$\mathcal{L}_N(t) = \mathbb{H} + \sum_{j=1}^{3} R_N^{(j)}(t)$$

with

$$\mathbb{H} = d\Gamma\left(-\Delta + v * |\varphi_t|^2 + q_t K_1 q_t\right)$$
$$+ \frac{1}{2} \int v(x-y) \left[\varphi_t(x) \varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + h.c.\right] dx dy$$

and

$$R_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma\left(q_t [v * |\varphi_t|^2 + K_{1,t}]\, q_t\right) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b\left(q_t [v * |\varphi_t|^2]\, \varphi_t\right) + h.c.$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}} \int v(x-y)\, \varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx dy + h.c.$$

$$R_{N,3}(t) = \frac{1}{2N} \int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx dy$$

STEP 2 :

GOAL : $\boxed{|\langle \Psi, [\mathcal{N}, \mathcal{L}_N(t)]\Psi\rangle| \leq C\,\langle \Psi, (\mathcal{N}+1)\Psi\rangle \quad \text{f. } \Psi \in \mathcal{F}_{\perp\varphi_t}^{\leq N}}$

$$[\mathcal{N}, \mathcal{L}_N(t)] = [\mathcal{N}, \mathbb{H}] + [\mathcal{N}, \sum_{j=1}^{3} R_N^{(j)}(t)]$$

with

$$\mathbb{H} = d\Gamma(-\Delta + v*|\varphi_t|^2 + q_t K_1 q_t)$$
$$+ \frac{1}{2}\int v(x-y)\left[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + h.c.\right] dx\,dy$$

and

$$R_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma(q_t[v*|\varphi_t|^2 + K_{1,t}]q_t) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b(q_t[v*|\varphi_t|^2]\varphi_t) + h.c.$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}}\int v(x-y)\,\varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\,dy + h.c.$$

$$R_{N,3}(t) = \frac{1}{2N}\int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\,dy$$

STEP 2 :

GOAL : $\boxed{ |\langle \Psi, [\mathcal{N}, \mathcal{L}_N(t)] \Psi \rangle| \leq C \langle \Psi, (\mathcal{N}+1) \Psi \rangle \quad \text{f.} \ \Psi \in \mathcal{F}_{\perp \varphi_t}^{\leq N} }$

$$[\mathcal{N}, \mathcal{L}_N(t)] = [\mathcal{N}, \mathbb{H}] + [\mathcal{N}, \sum_{j=1}^{3} R_N^{(j)}(t)]$$

with

$$[\mathcal{N}, \mathbb{H}] = [\mathcal{N}, d\Gamma(-\Delta + v * |\varphi_t|^2 + q_t K_1 q_t)]$$
$$+ [\mathcal{N}, \tfrac{1}{2} \int v(x-y) [\varphi_t(x) \varphi_t(y)\, b^*(q_{t,x}) b^*(q_{t,y}) + \text{h.c.}]\, dx dy]$$

and

$$R_{N,1}(t) = \frac{1 - \mathcal{N}_+(t)}{2N} d\Gamma(q_t [v * |\varphi_t|^2 + K_{1,t}] q_t) + \frac{\mathcal{N}_+(t)}{\sqrt{N}} b(q_t [v * |\varphi_t|^2] \varphi_t) + \text{h.c.}$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}} \int v(x-y)\, \varphi_t(y)\, a^*(q_{t,x}) a(q_{t,x})\, b(q_{t,y})\, dx dy + \text{h.c.}$$

$$R_{N,3}(t) = \frac{1}{2N} \int v(x-y)\, a^*(q_{t,x}) a^*(q_{t,y})\, a(q_{t,x}) a(q_{t,y})\, dx dy$$

STEP (2) :

GOAL : $\boxed{\left|\langle \Psi, [\mathcal{N}, \mathcal{L}_N(t)]\Psi\rangle\right| \leq C\,\langle \Psi, (\mathcal{N}+1)\Psi\rangle \quad \text{f. } \Psi \in \mathcal{F}_{\perp \varphi_t}^{\leq N}}$

$$[\mathcal{N}, \mathcal{L}_N(t)] = [\mathcal{N}, \mathbb{H}] + \Big[\mathcal{N}, \sum_{j=1}^{3} R_N^{(j)}(t)\Big]$$

with

$$[\mathcal{N}, \mathbb{H}] = \cancel{[\mathcal{N}, d\Gamma(-\Delta + v*|\varphi_t|^2 + q_t K_1 q_t)]}$$
$$+ \Big[\mathcal{N}, \tfrac{1}{2}\int v(x-y)\big[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + \text{h.c.}\big]\, dx\, dy\Big]$$

and

$$R_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma\big(q_t[v*|\varphi_t|^2 + K_{1,t}]q_t\big) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b\big(q_t[v*|\varphi_t|^2]\varphi_t\big) + \text{h.c.}$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}}\int v(x-y)\,\varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\, dy + \text{h.c.}$$

$$R_{N,3}(t) = \frac{1}{2N}\int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\, dy$$

STEP ②:

GOAL: $\boxed{|\langle \Psi, [\mathcal{N}, \mathcal{L}_N(t)] \Psi\rangle| \leq C \langle \Psi, (\mathcal{N}+1)\Psi\rangle \quad \text{f. } \Psi \in \mathcal{F}_{\perp\varphi_t}^{\leq N}}$

$$[\mathcal{N}, \mathcal{L}_N(t)] = [\mathcal{N}, \mathbb{H}] + \Big[\mathcal{N}, \sum_{j=1}^{3} R_N^{(j)}(t)\Big]$$

$[\mathcal{N}, b^*(h)] = b^*(h)$

$[\mathcal{N}, b(h)] = -b(h)$

with

$$[\mathcal{N}, \mathbb{H}] = \cancel{[\mathcal{N}, d\Gamma(-\Delta + v*|\varphi_t|^2 + q_t K_1 q_t)]}$$
$$+ \Big[\mathcal{N}, \tfrac{1}{2}\int v(x-y)\big[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) + h.c.\big]\, dx\,dy\Big]$$

and

$$R_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma(q_t[v*|\varphi_t|^2 + K_{1,t}]q_t) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b(q_t[v*|\varphi_t|^2]\varphi_t) + h.c.$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}} \int v(x-y)\, \varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\,dy + h.c.$$

$$R_{N,3}(t) = \frac{1}{2N} \int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\,dy$$

STEP 2 :

GOAL : $$\boxed{|\langle \Psi, [\mathcal{N}, \mathcal{L}_N(t)]\Psi\rangle| \le C\, \langle \Psi, (\mathcal{N}+1)\Psi\rangle \quad \text{f. } \Psi \in \mathcal{F}_{\perp \varphi_t}^{\le N}}$$

$$[\mathcal{N}, \mathcal{L}_N(t)] = [\mathcal{N}, \mathbb{H}] + [\mathcal{N}, \sum_{j=1}^{3} R_N^{(j)}(t)]$$

$$[\mathcal{N}, b^*(h)] = b^*(h)$$

$$[\mathcal{N}, b(h)] = -b(h)$$

with

$$[\mathcal{N}, \mathbb{H}] = \cancel{[\mathcal{N}, d\Gamma(-\Delta + v*|\varphi_t|^2 + q_t K_1 q_t)]}$$
$$- \int v(x-y)\big[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) - h.c.\big]\, dx\,dy$$

and

$$R_{N,1}(t) = \frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma\big(q_t[v*|\varphi_t|^2 + K_{1,t}]q_t\big) + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b\big(q_t[v*|\varphi_t|^2]\varphi_t\big) + h.c.$$

$$R_{N,2}(t) = \frac{1}{\sqrt{N}} \int v(x-y)\, \varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\,dy + h.c.$$

$$R_{N,3}(t) = \frac{1}{2N} \int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\,dy$$

STEP 2 :

GOAL: $\boxed{|\langle \Psi, [\mathcal{N}, \mathcal{L}_N(t)]\Psi\rangle| \le C \langle \Psi, (\mathcal{N}+1)\Psi\rangle \quad \text{f. } \Psi \in \mathcal{F}_{\perp\varphi_t}^{\le N}}$

$$[\mathcal{N}, \mathcal{L}_N(t)] = [\mathcal{N}, \mathbb{H}] + \Big[\mathcal{N}, \sum_{j=1}^{3} R_N^{(j)}(t)\Big]$$

$$[\mathcal{N}, b^*(h)] = b^*(h)$$

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$$[\mathcal{N}, \mathbb{H}] = \cancel{[\mathcal{N}, d\Gamma(-\Delta + v*|\varphi_t|^2 + q_t K_1 q_t)]}$$
$$- \int v(x-y)\big[\varphi_t(x)\varphi_t(y)\, b^*(q_{t,x})\, b^*(q_{t,y}) - h.c.\big]\, dx\,dy$$

and

$$[\mathcal{N}, R_{N,1}(t)] = \cancel{\frac{1-\mathcal{N}_+(t)}{2N}\, d\Gamma\big(q_t[v*|\varphi_t|^2 + K_{1,t}]q_t\big)} + \frac{\mathcal{N}_+(t)}{\sqrt{N}}\, b\big(q_t[v*|\varphi_t|^2]\varphi_t\big) - h.c.$$

$$[\mathcal{N}, R_{N,2}(t)] = \frac{1}{\sqrt{N}} \int v(x-y)\, \varphi_t(y)\, a^*(q_{t,x})\, a(q_{t,x})\, b(q_{t,y})\, dx\,dy - h.c.$$

$$[\mathcal{N}, R_{N,3}(t)] = \cancel{\frac{1}{2N} \int v(x-y)\, a^*(q_{t,x})\, a^*(q_{t,y})\, a(q_{t,x})\, a(q_{t,y})\, dx\,dy}$$

$\rightsquigarrow$ blackboard.

4.4 SUMMARY

- Bound on excitations

$$\mathcal{W}_N(t;0)\,\mathcal{N}\,\mathcal{W}_N^*(t;0) \leq e^{C_1|t|}\,C_2$$

excitations

fluctuation dynamics

4.4 SUMMARY

- Bound on excitations

$$\mathcal{W}_N(t;0)\,\mathcal{N}\,\mathcal{W}_N(t;0) \leq e^{C_1|t|}\,C_2$$

implies for Random variables $\{Y_i^{N,t}\}$

$$\mathbb{P}_{\Psi_{N,t}}\left[\,\left|\frac{1}{N}\sum_{i=1}^{N} Y_i^{N,t} - \langle\varphi_t, O^{(1)}\varphi_t\rangle\right| > \delta\right] \xrightarrow{N\to\infty} 0$$

solution of
$i\partial_t \Psi_{N,t} = H_N \Psi_{N,t}$

solution to
$i\partial_t \varphi_t = (-\Delta + v * |\varphi_t|^2)\varphi_t$

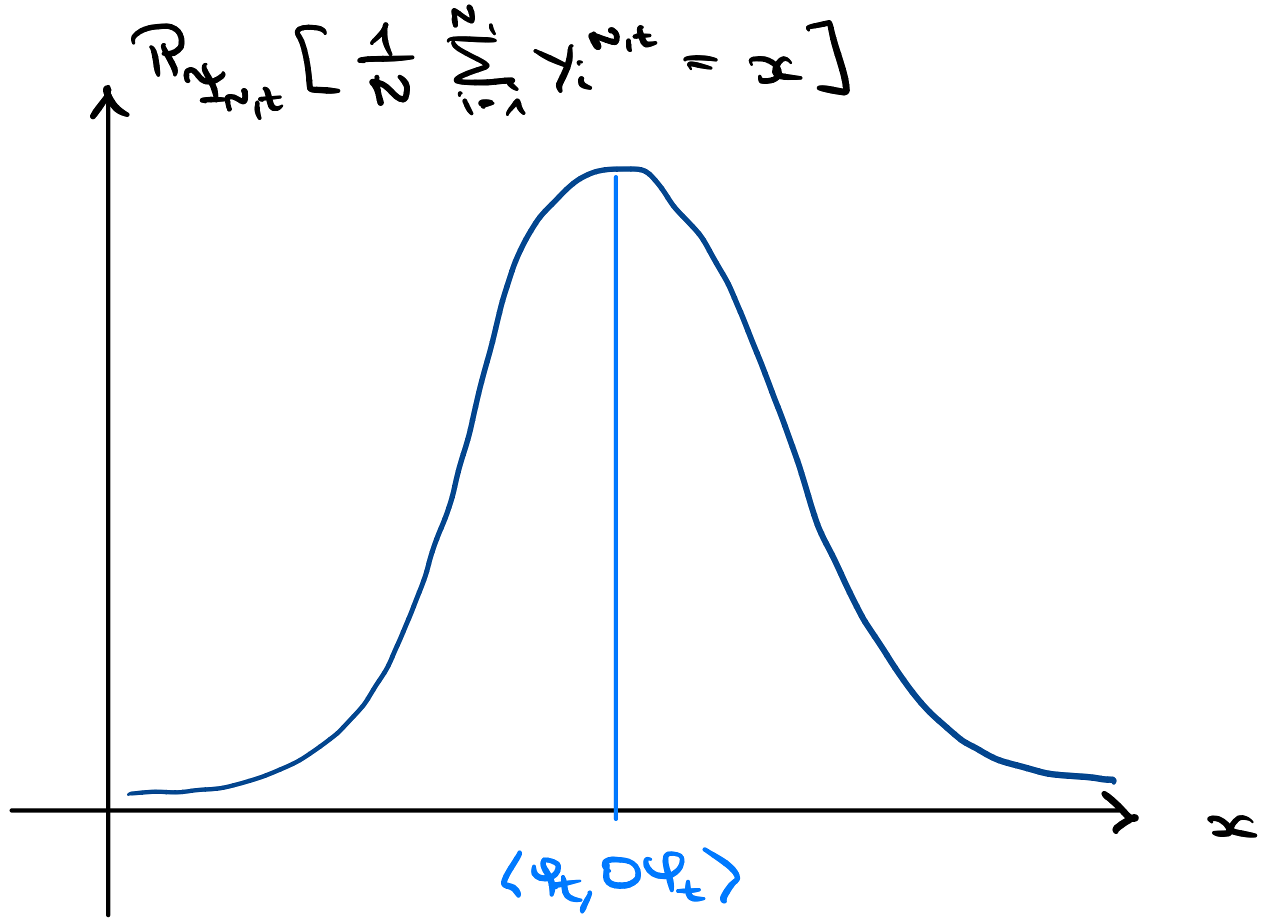
4.4 SUMMARY

- Bound on excitations

$$W_N(t;0)\, \mathcal{N}\, W_N(t;0) \le e^{C_1|t|}\, C_2$$

implies for Random variables $\{Y_i^{N,t}\}$

$$\mathbb{P}_{\Psi_{N,t}}\left[\, \left| \frac{1}{N} \sum_{i=1}^{N} Y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t \rangle \right| > \delta \right] \xrightarrow{N\to\infty} 0$$



4.4 SUMMARY

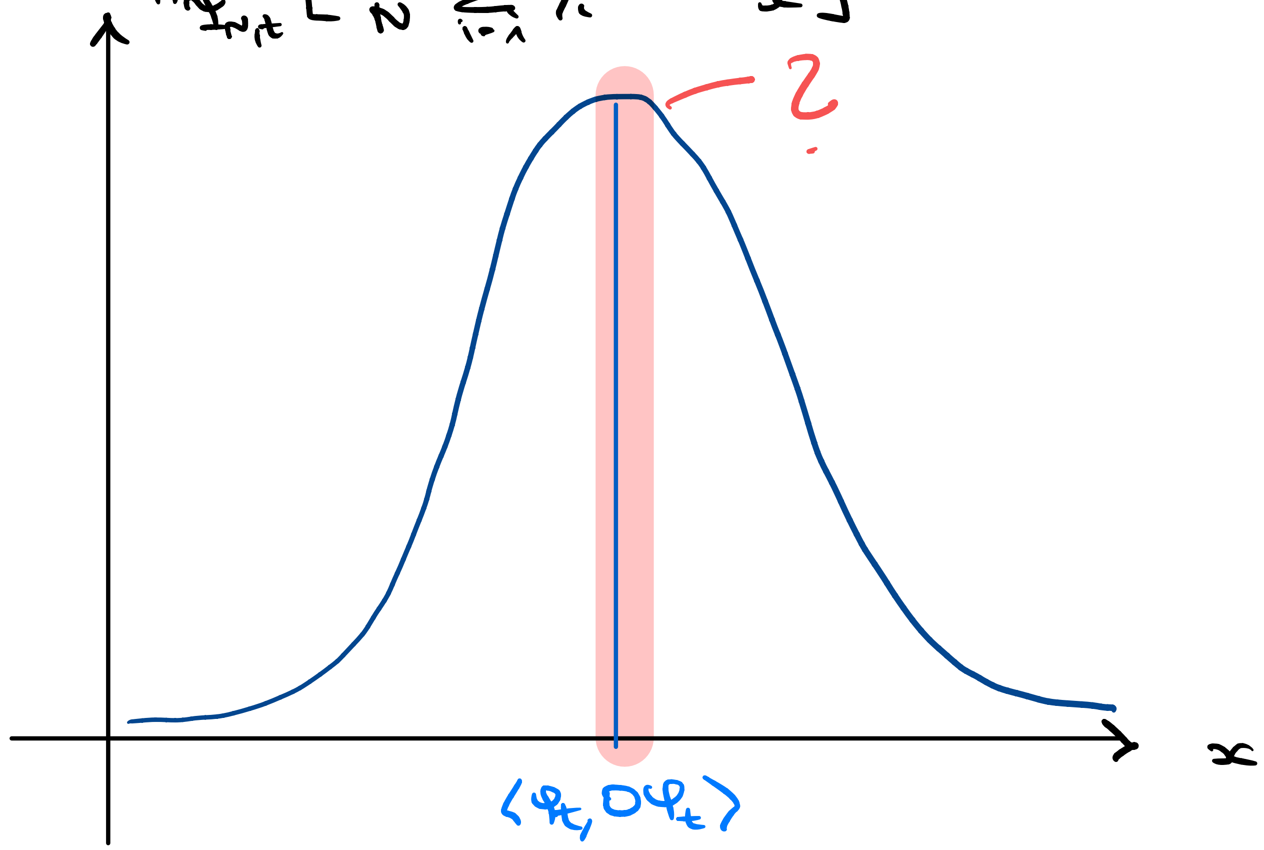
- Bound on excitations

$$\mathcal{W}_N(t;0)\, \mathcal{N}\, \mathcal{W}_N(t;0) \leq e^{C_1|t|}\, C_2$$

implies for Random variables $\{Y_i^{N,t}\}$

$$\mathbb{P}_{\Psi_{N,t}}\left[\left| \frac{1}{N} \sum_{i=1}^{N} Y_i^{N,t} - \langle \varphi_t, O^{(1)} \varphi_t \rangle \right| > \delta \right] \xrightarrow{N \to \infty} 0$$

$$\mathbb{P}_{\Psi_{N,t}}\left[\frac{1}{N} \sum_{i=1}^{N} Y_i^{N,t} = x \right]$$



NEXT: characterization of fluctuations through CLT